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A COMPUTER AIDED MULTIVARIABLE CONTROL SYSTEMS DESIGN TECHNIQUE WITH APPLICATION TO AIRCRAFT FLYING QUALITIES

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A computer-aided multivariable control system design technique is developed for matching the frequency response of a compensated closed loop system with a given 'desired' transfer function matrix over a frequency interval of interest. Two examples of application of the technique are provided as well as a simple robustness analysis of one of the application examples.

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Thanks are due to my graduate student, Mukhtar Shabbarhussain who, under my guidance, carried out complicated derivations of the design equations. We have also written the code in Fortran 77 for the computer implementation of the algorithm which is available with Mr. Tom Gentry, the Project Engineer. It is also available with the Principle Investigator. My other graduate student, Seong Kim, worked on the project toward the final phase and I thank him for his contribution.

Sincere thanks are also due to Nalda Blair, our department secretary, who has done an excellent job of typing this report.

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1. INTRODUCTION

The main objective of this project is to develop a computer–aided multivariable control system design technique by matching the frequency responses of the compensated closed loop system and a given 'desired' transfer function matrix over a frequency interval of interest. The motivation for this approach is to come up 'quickly' with a candidate for the compensation which could be 'fine tuned' for practical applications.

The proposed MIMO (multi–input multi–output) control design technique is a generalization of the SISO (single–input single–output) technique of the Principal Investigator [13]. An aircraft application of the SISO technique was presented by Gentry and Pujara in [4].

Under the MIMO design technique presented in this report, one starts with a transfer function matrix of a stable plant and the objective is to design a MIMO controller so that the overall closed loop transfer function matrix is 'close' to a given desired transfer function matrix. The desired transfer function matrix is synthesized to incorporate a given set of specifications. As was mentioned earlier, the plant has to be stable and if it is unstable, then it has to be stabilized before applying the proposed technique.

Under the proposed technique, the controller parameters are obtained by minimizing a weighted mean square error between the frequency responses of the compensated closed loop system and a desired closed loop system. The weighting function in the error mentioned above is chosen in such a way that the accuracy of the error is maintained as far as possible and in addition to this, the parameters of the controller turn out to be solutions of linear algebraic equations. This last aspect of the algorithm is extremely important in as much as this leads to a simple computer–aided package to implement the algorithm.

The program to implement the algorithm has been written in Fortran 77 and is quite simple to use. All that the user has to do is to input the coefficients of the plant

transfer function matrix, the limits of frequency interval and the order of the controller. These frequency intervals are identified by plotting the magnitude response of the components of the desired transfer function matrix. As output, the user gets the parameters of the controller, the frequency responses of the compensated closed loop system and the desired transfer function matrix. If this is satisfactory, one is done. If not, then the user increases the order of the controller and/or 'tinkers' with the limits of integration for the error function until satisfactory results are obtained. This 'trial and error' is not excessively time consuming because it is a computer–aided technique. Four examples are solved in this report for illustration with excellent results. Example 1 is the same example studied by Chen [1] and the comparison of the results by the proposed technique with those of Chen [1] shows that the proposed technique gives letter results. The three parts of the second example are related to YF–16 CCV.

In addition to this, a modest 'robustness' analysis of the design for the YF–16 CCV control system design obtained by the proposed technique in Example 2 has been carried out. Only one parameter in the plant after it was stabilized was varied. By using the same controller as obtained for the nominal plant, it was found that for a larger range of values for this parameter, the inner loop maintains its stability and the overall closed loop system also maintains an excellent match with the desired closed loop transfer function matrix. This has been quantified by drawing a graph of the error function (between the frequency responses of the compensated system and the desired system over the frequency interval [0,10] rad.) against the changes in the parameter. The frequency interval [0,10] rad/sec was picked as this interval is of great significance from the point of aircraft flying qualities. In addition to this, some sample graphs (over the range of the parameter) for magnitude, phase and time response have been given which portray good match and reinforce the results predicted by the error function. Evidently, a lot more time and effort is required to do a complete 'robustness' analysis for the example under consideration here. But this is a reasonably good start.

It should be pointed out that the design technique proposed in this report is only for nominal plant and no disturbance–rejection or noise–suppression considerations were included in the error criterion. It is a complicated matter and the inclusion of these concerns in the design methodology is left for future research work.

A program written in Fortran 77 in the form of a floppy diskette has already been submitted to Mr. Tom Gentry, the Project Engineer.

2. A BRIEF REVIEW OF THE FREQUENCY MATCHING TECHNIQUE FOR SINGLE–INPUT SINGLE–OUTPUT CONTROL SYSTEMS

2.1 Introduction

The objective of this chapter is to review the frequency matching technique for the SISO (single–input single–output) control system. This technique uses the idea of a weighting function to 'linearize' the problem, which has been in the literature for quite some time. This idea was originally used by Levy [10] for complex curve fitting, and then used by several researchers [14,15,16,17,18,19] for model reduction problems and digitizing continuous systems. In addition to this linearizing idea, we have added several other features in the error criterion to make it a viable and practical design tool. Under this technique, the controller parameters are obtained by matching the frequency responses of the compensated closed loop system and a 'desired' closed loop system. The 'desired' transfer function is synthesized to meet a given set of specifications, which for example could include flying qualities criteria for an aircraft control system design.

The error is defined in such a way that the controller parameters turn out to be solutions of linear algebraic equations. This is quite significant [2,6] when contrasted with the results where the controller parameters turn out to be solutions of nonlinear algebraic equations which, in general, are harder to solve.

2.2 The SISO Frequency Matching Technique

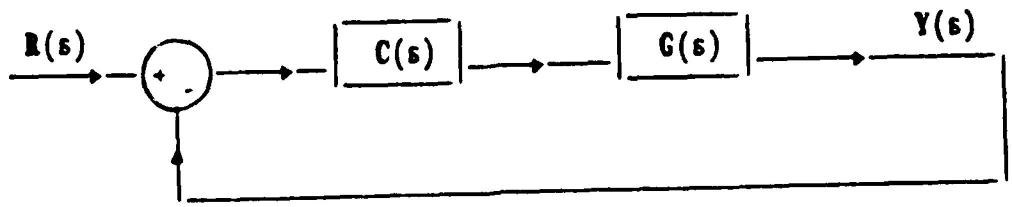

Figure 2.1: Block diagram for SISO control systems

Consider a typical unit feedback control system as represented by its block diagram as shown in Figure 2.1 where G(s) is the plant, assumed stable, and C(s) is a controller. It is desired to synthesize a controller C(s) so that the compensated closed loop transfer function is 'close' to a certain 'desired' closed loop transfer function over a frequency interval of interest. Suppose D(s) is a 'desired' closed loop transfer function which has been synthesized to satisfy a given set of specifications. Suppose

$$C(s) = \frac{\sum_{i=0}^{m} a_i s^i}{1+\sum_{i=1}^{n} b_i s^i}, \quad m \leq n \tag{2.1}$$

where a's and b's are unknown real parameters. Define

$$G(j\omega) = \frac{L_1(\omega)+jL_2(\omega)}{M_1(\omega)+jM_2(\omega)} \tag{2.2}$$

$$D(j\omega) = \frac{N_1(\omega)+jN_2(\omega)}{P_1(\omega)+jP_2(\omega)} \tag{2.3}$$

where $L_1(\omega)$, $L_2(\omega)$, $M_1(\omega)$, $M_2(\omega)$, $N_1(\omega)$, $N_2(\omega)$, $P_1(\omega)$ and $P_2(\omega)$ are polynomials in ω. Then the frequency response of the compensated closed loop transfer function can be written as

$$F(j\omega) = \frac{\left\{\left[\frac{\sum_{i=0}^{m} a_i (j\omega)^i}{1+\sum_{i=1}^{n} b_i (j\omega)^i}\right]\left[\frac{L_1(\omega) + jL_2(\omega)}{M_1(\omega) + jM_2(\omega)}\right]\right\}}{\left\{1 + \left[\frac{\sum_{i=0}^{m} a_i (j\omega)^i}{1+\sum_{i=1}^{n} b_i (j\omega)^i}\right]\left[\frac{L_1(\omega) + jL_2(\omega)}{M_1(\omega) + jM_2(\omega)}\right]\right\}} \tag{2.4}$$

As in [13], the weighted mean square error between the frequency responses of F(jω) and D(jω) can be written as

$$\begin{aligned} E \;=\; \int_{\omega_1}^{\omega_2} \Bigg| & [P_1(\omega) + jP_2(\omega)][L_1(\omega) + jL_2(\omega)] \left[\sum_{i=0}^{m} a_i (j\omega)^i\right] \\ & -[N_1(\omega) + jN_2(\omega)] \left\{ [M_1(\omega) + jM_2(\omega)] \left[1 + \sum_{i=0}^{m} b_i (j\omega)^i\right] \right. \\ & \left. +[L_1(\omega) + jL_2(\omega)] \left[\sum_{i=0}^{m} a_i (j\omega)^i\right] \right\} \Bigg|^2 d\omega \end{aligned} \tag{2.5}$$

where $[\omega_1, \omega_2]$ is the frequency interval of interest.

As in [13], the minimization of E with respect to the controller parameters, results in linear algebraic equations involving a's and b's. These equations are written as a system of equations in matrix form as was described in [13]. The complete derivation and application of this frequency matching technique for the SISO control systems algorithm can be found in [13] and [4], respectively.

2.3 Summary

In this chapter, a computer–aided technique of designing a SISO control system is reviewed. The controller–parameters are obtained by matching the frequency responses of the compensated closed loop system and a 'desired' closed loop system over a certain frequency interval of interest. The technique is easy to simulate as the controller parameters turn out to be solutions of linear algebraic equations. A much more detailed explanation of the SISO frequency matching technique can be found in [13]. In addition to this, [4,13] is another good reference as an illustration of potential applications of the technique for aircraft control systems design.

3. MULTIVARIABLE FREQUENCY MATCHING TECHNIQUE

The objective of this chapter is to generalize the frequency matching technique for the SISO control system design case to the MIMO case. We begin with a brief introduction to the multivariable function matching technique algorithm. Then we do the problem formulation and then give the complete derivation of the algorithm. Finally, we give some explanatory remarks regarding the application of the technique.

3.1 Introduction

The main thrust of this report is to develop a computer–aided control system design technique for linear multivariable continuous control systems via frequency–matching. As in the SISO case, this is accomplished by matching the frequency responses between the compensated closed loop system and a 'desired' closed loop system by means of a weighted mean square error. A suitable generalization of the error function for the MIMO case would be defined to obtain the parameters of the controller.

We propose to obtain the controller parameters by minimizing a weighted mean square error between the frequency responses of the compensated closed loop transfer function matrix and the 'desired' closed loop transfer function matrix over a certain frequency interval of interest. The 'desired' closed loop transfer function matrix is synthesized to meet a given set of specifications. A similar idea of using a 'desired' transfer function matrix for multivariable control system design has been used by Lehtomaki, Stein and Walls [9], Chen [1], etc. As in the SISO case, the error is defined in such a way that the controller parameters turn out to be solutions of linear algebraic equations. To make the technique fairly general, intervals of frequency for different elements of the transfer function matrix could be used. The derivations were quite involved and intricate, but eventually quite useful.

3.2 Problem Formulation

Consider a typical multivariable feedback control system of order n represented by the following block diagram

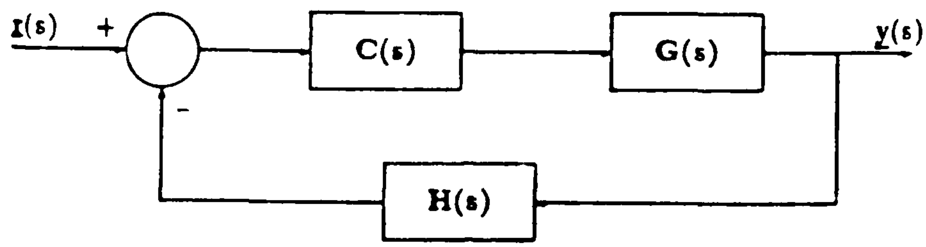

Figure 3.1: Multivariable feedback control system

where

$\underline{r}(s)$ = vector of reference inputs, of order n.

$\underline{y}(s)$ = vector of outputs, of order n.

$\underline{G}(s)$ = n x n transfer function matrix of the plant.

H(s) = n x n transfer function matrix of the feedback element.

C(s) = n x n transfer function matrix of the controller

which has to be synthesized to meet the design specifications.

The closed loop transfer function matrix F(s) for this system is given by

$$\mathbf{F}(s) = [\mathbf{I}_n + \mathbf{G}(s)\mathbf{C}(s)\mathbf{H}(s)]^{-1}\,\mathbf{G}(s)\mathbf{C}(s) \tag{3.1}$$

where I_n is the identity matrix of order n. Suppose D(s) is a 'desired' n x n closed loop transfer function matrix which has been synthesized to satisfy a given set of specifications. These, for example, could include flying qualities criteria for aerospace application. Suppose

$$\mathbf{G}(s) = \frac{[g_{ik}(s)]}{g(s)}\ , \quad \mathbf{H}(s) = \frac{[h_{ik}(s)]}{h(s)}\ , \tag{3.2}$$

$$\mathbf{C}(s) = \frac{[c_{ik}(s)]}{\alpha(s)}\ , \quad \mathbf{D}(s) = \frac{[d_{ik}(s)]}{d(s)}\ , \tag{3.3}$$

and $$c_{ik}(s) = \sum_{r=0}^{p(i,k)} a_r^{(i,k)} s^r \ , \quad \alpha(s) = 1 + \sum_{r=1}^{q} b_r s^r \ , \quad \begin{array}{l} p(i,k) \leq q \ , \\ 1 < i, k < n \end{array} \tag{3.4}$$

where g's, h's, c's. d's and $\alpha(s)$ are polynomials in s with real coefficients. The controller C(s) has a common denominator $\alpha(s)$, with $\alpha_r^{(i,k)}$ and b_z, for $0 \leq r \leq p(i,k)$ and $1 \leq i, k \leq n$, and $1 \leq z \leq q$, respectively, as the real unknown parameters. In this chapter, a notation of the form $[g_{ik}(s)]$ will stand for an n x n matrix with $g_{ik}(s)$ as its $(i,k)^{th}$ entry.

Now we define a mean square error between the frequency responses $\mathbf{F}(j\omega)$ of the compensated closed loop system and $\mathbf{D}(j\omega)$ of the 'desired' closed loop system, to determine the unknown parameters in the controller. This is defined as follows

$$E(\underline{a}, \underline{b}) = \sum_{i=1}^{n} \sum_{k=1}^{n} \int_{\omega_{ik}}^{\omega'_{ik}} |F_{ik}(j\omega) - D_{ik}(j\omega)|^2 \, d\omega \tag{3.5}$$

where $F_{ik}(j\omega)$ and $D_{ik}(j\omega)$ are the $(i,k)^{th}$ elements of the matrices $F(j\omega)$ and $D(j\omega)$, respectively. Also $[\omega_{ik}, \omega'_{ik}]$ is the frequency interval of interest for the $(i,k)^{th}$ element in the 'desired' transfer function matrix D(s) which usually is the bandwidth of that particular transfer function. Also $\underline{a}$ and $\underline{b}$ are the vectors of the unknown controller parameters, $\alpha_r^{(i,k)}$ for $0 \leq r \leq p(i,k)$, $1 \leq i,k \leq n$, and b_z for $1 \leq z \leq q$, respectively. An optimal set of $\{\alpha_r^{(i,k)}, b_z\}$ for $0 \leq r \leq p(i,k)$, $1 \leq i, k \leq n$, and $1 \leq z \leq q$, can be obtained by minimizing $E(\underline{a},\underline{b})$.

It can easily be checked that $F_{ik}(j\omega)$ is the $(i,k)^{th}$ element of

$$\left[\frac{g(j\omega)h(j\omega)\alpha(j\omega)[\delta_{ik}] + [g_{ik}(j\omega)][c_{ik}(j\omega)][h_{ik}(j\omega)]}{g(j\omega)h(j\omega)\alpha(j\omega)}\right]^{-1} \frac{[g_{ik}(j\omega)][c_{ik}(j\omega)]}{g(j\omega)\alpha(j\omega)} \tag{3.6}$$

where δ_{ik} is the Kronecker delta. Now to obtain the controller parameters, we minimize the error function $E(\underline{a},\underline{b})$ as in equation (3.5) by equating to zero the partial derivatives of $E(\underline{a},\underline{b})$ with respect to $\alpha_r^{(i,k)}$ and b_z, for $0 \leq r \leq p(i,k)$, $1 \leq i, k \leq n$, and $1 \leq z \leq q$. This way one will get a set of complicated nonlinear algebraic equations.

3.3 The Algorithm

We will now show that if the error E($\underline{a}$,$\underline{b}$) is suitably modified, while retaining a reasonable degree of accuracy of approximation, the multivariable frequency–matching technique will lead only to solving of much simpler linear algebraic equations involving the controller parameters. We feel that this is a key point in the development that follows. We want the frequency response of the compensated closed loop transfer function matrix, F(jω) to match the frequency response of the 'desired' closed loop transfer function matrix, *D(jω) over a frequency interval of interest. In symbols, we wish to achieve*

$$[\mathbf{I}_n + \mathbf{G}(j\omega)\mathbf{C}(j\omega)\mathbf{H}(j\omega)]^{-1}\,\mathbf{G}(j\omega)\mathbf{C}(j\omega) \approx \mathbf{D}(j\omega) \tag{3.7}$$

or

$$\mathbf{G}(j\omega)\mathbf{C}(j\omega) \approx [\mathbf{I}_n + \mathbf{G}(j\omega)\mathbf{C}(j\omega)\mathbf{H}(j\omega)]\,\mathbf{D}(j\omega) \tag{3.8}$$

for all ω in the frequency interval of interest.

Making appropriate substitution, we thus wish to achieve

$$\begin{aligned} &\frac{[g_{ik}(j\omega)][c_{ik}(j\omega)]}{g(j\omega)\alpha(j\omega)} \\ &\approx \left[\frac{g(j\omega)h(j\omega)\alpha(j\omega)[\delta_{ik}] + [g_{ik}(j\omega)][c_{ik}(j\omega)][h_{ik}(j\omega)]}{g(j\omega)h(j\omega)\alpha(j\omega)}\right]\frac{[d_{ik}(j\omega)]}{d(j\omega)} \end{aligned} \tag{3.9}$$

Now we cross multiply in the above Equation (3.9). Thus we see that we wish to achieve

$$\begin{aligned} &[g_{ik}(j\omega)][c_{ik}(j\omega)]h(j\omega)d(j\omega) \\ &\approx\ [g(j\omega)h(j\omega)\alpha(j\omega)[\delta_{ik}] + [g_{ik}(j\omega)][c_{ik}(j\omega)][h_{ik}(j\omega)]]\,[d_{ik}(j\omega)] \end{aligned} \tag{3.10}$$

for all ω in the frequency interval of interest. This has, in some sense, a 'linearizing effect' on the minimization of E($\underline{a}$,$\underline{b}$) without losing much accuracy. Keeping in view the above intuitive idea, the modified error E_m($\underline{a}$,$\underline{b}$) is defined as

$$E_m(\underline{a},\underline{b}) = \sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\omega_{ik}}^{\omega'_{ik}} |h(j\omega)d(j\omega)l_{ik}(j\omega) - (g(j\omega)h(j\omega)\alpha(j\omega)d_{ik}(j\omega) + n_{ik}(j\omega))|^2\, d\omega \tag{3.11}$$

where

$$\left.\begin{aligned} l_{ik}(j\omega) &= \sum_{r=1}^{n} g_{ir}(j\omega)c_{rk}(j\omega)\ , \\ m_{ik}(j\omega) &= \sum_{r=1}^{n} h_{ir}(j\omega)d_{rk}(j\omega)\ , \\ n_{ik}(j\omega) &= \sum_{r=1}^{n} l_{ir}(j\omega)m_{rk}(j\omega)\ ,\ \ 1 \le i,k \le n \end{aligned}\right\} \tag{3.12}$$

The above 'linearization' approach for SISO system design and model reduction has been effectively used by several researchers, including Levy [10], Rao and Lamba [20], Rattan [18], Rattan and Yeh [19], Pujara and Rattan [15,16,17], and Pujara [13,14]. For the sake of brevity, from now on, we will write $E_m(\underline{a},\underline{b})$ as just E_m.

Before we begin to take the partial derivatives of the error function E_m with respect to the unknown controller parameters and set it equal to zero, we need to simplify E_m. First, define the frequency responses of the $(i,k)^{ik}$ entries of the plant, the feedback element, the 'desired' and the controller transfer function matrices as follows

$$\left.\begin{aligned} g_{ik}(j\omega) &= A_{ik}(\omega) + jB_{ik}(\omega)\ ,\quad g(j\omega) = A(\omega) + jB(\omega) \\ h_{ik}(j\omega) &= E_{ik}(\omega) + jF_{ik}(\omega)\ ,\quad h(j\omega) = E(\omega) + jF(\omega) \\ d_{ik}(j\omega) &= L_{ik}(\omega) + jM_{ik}(\omega)\ ,\quad d(j\omega) = L(\omega) + jM(\omega) \end{aligned}\right\} \tag{3.13}$$

and

$$c_{ik}(j\omega) = \sum_{r=0}^{p(i,k)} a_r^{(i,k)}(j\omega)^r = \sigma_{ik}(\omega) + j\beta_{ik}(\omega) \tag{3.14}$$

$$\alpha(j\omega) = 1 + \sum_{r=1}^{q} b_r(j\omega)^r = \sigma(\omega) + j\beta(\omega) \tag{3.15}$$

where,

$$
\left.\begin{aligned}
\sigma_{ik}(\omega) &= a_0^{(i,k)} - a_2^{(i,k)}\omega^2 + a_4^{(i,k)}\omega^4 - a_6^{(i,k)}\omega^6 + \cdots \\
\beta_{ik}(\omega) &= a_1^{(i,k)}\omega - a_3^{(i,k)}\omega^3 + a_5^{(i,k)}\omega^5 - a_7^{(i,k)}\omega^7 + \cdots \\
\sigma(\omega) &= 1 - b_2\omega^2 + b_4\omega^4 - b_6\omega^6 + \cdots \\
\beta(\omega) &= b_1\omega - b_3\omega^3 + b_5\omega^5 - b_7\omega^7 + \cdots
\end{aligned}\right\} \tag{3.16}
$$

Using the above notation, we define the following:

$$
\left.\begin{aligned}
\psi(\omega) &= \Re\{h(j\omega)d(j\omega)\} = E(\omega)L(\omega) - F(\omega)M(\omega) \\
\theta(\omega) &= \Im\{h(j\omega)d(j\omega)\} = E(\omega)M(\omega) + F(\omega)L(\omega) \\
\phi(\omega) &= \Re\{g(j\omega)h(j\omega)\} = A(\omega)E(\omega) - B(\omega)F(\omega) \\
\gamma(\omega) &= \Im\{g(j\omega)h(j\omega)\} = A(\omega)F(\omega) + B(\omega)E(\omega)
\end{aligned}\right\} \tag{3.17}
$$

and also

$$
Q_{ik}(\omega) = \Re\{l_{ik}(j\omega)\} = \Re\left\{\sum_{r=1}^{n} g_{ir}(j\omega)c_{rk}(j\omega)\right\} \tag{3.18}
$$

$$
= \sum_{r=1}^{n}[A_{ir}(\omega)\sigma_{rk}(\omega) - B_{ir}(\omega)\beta_{rk}(\omega)] \tag{3.19}
$$

$$
R_{ik}(\omega) = \Im\{l_{ik}(j\omega)\} = \Im\left\{\sum_{r=1}^{n} g_{ir}(j\omega)c_{rk}(j\omega)\right\} \tag{3.20}
$$

$$
= \sum_{r=1}^{n}[A_{ir}(\omega)\beta_{rk}(\omega) + B_{ir}(\omega)\sigma_{rk}(\omega)] \tag{3.21}
$$

$$
\psi_{ik}(\omega) = \Re\{m_{ik}(j\omega)\} = \Re\left\{\sum_{r=1}^{n} h_{ir}(j\omega)d_{rk}(j\omega)\right\} \tag{3.22}
$$

$$
= \sum_{r=1}^{n}[E_{ir}(\omega)L_{rk}(\omega) - F_{ir}(\omega)M_{rk}(\omega)] \tag{3.23}
$$

$$
\theta_{ik}(\omega) = \Im\{m_{ik}(j\omega)\} = \Im\left\{\sum_{r=1}^{n} h_{ir}(j\omega)d_{rk}(j\omega)\right\} \tag{3.24}
$$

$$
= \sum_{r=1}^{n}[E_{ir}(\omega)M_{rk}(\omega) + F_{ir}(\omega)L_{rk}(\omega)] \tag{3.25}
$$

where $\Re\{\cdot\}$ and $\Im\{\cdot\}$ stand for real and imaginary part of a function, respectively. Now, finally to simplify the modified Error E_m, we let

$$\begin{aligned}
\mu_{ik}(\omega) &= \Re\{[h(j\omega)d(j\omega)]l_{ik}(j\omega)\} && (3.26)\\
&= \Re\{[\psi(\omega) + j\theta(\omega)][Q_{ik}(\omega) + jR_{ik}(\omega)]\} && (3.27)\\
&= \psi(\omega)Q_{ik}(\omega) - \theta(\omega)R_{ik}(\omega) && (3.28)\\
&= \sum_{r=1}^{n}[(\psi(\omega)A_{ir}(\omega) - \theta(\omega)B_{ir}(\omega))\sigma_{rk}(\omega) \\
&\qquad -(\theta(\omega)A_{ir}(\omega) + \psi(\omega)B_{ir}(\omega))\beta_{rk}(\omega)] && (3.29)
\end{aligned}$$

$$\begin{aligned}
\lambda_{ik}(\omega) &= \Im\{[h(j\omega)d(j\omega)]l_{ik}(j\omega)\} && (3.30)\\
&= \Im\{[\psi(\omega) + j\theta(\omega)][Q_{ik}(\omega) + jR_{ik}(\omega)]\} && (3.31)\\
&= \theta(\omega)Q_{ik}(\omega) + \psi(\omega)R_{ik}(\omega) && (3.32)\\
&= \sum_{r=1}^{n}[(\theta(\omega)A_{ir}(\omega) + \psi(\omega)B_{ir}(\omega))\sigma_{rk}(\omega) \\
&\qquad +(\psi(\omega)A_{ir}(\omega) - \theta(\omega)B_{ir}(\omega))\beta_{rk}(\omega)] && (3.33)
\end{aligned}$$

$$\begin{aligned}
T_{ik}(\omega) &= \Re\{[g(j\omega)h(j\omega)]\alpha(j\omega)d_{ik}(j\omega)\} && (3.34)\\
&= \Re\{[\phi(\omega) + j\gamma(\omega)][\sigma(\omega) + j\beta(\omega)][L_{ik}(\omega) + jM_{ik}(\omega)]\} && (3.35)\\
&= [(\phi(\omega)L_{ik}(\omega) - \gamma(\omega)M_{ik}(\omega))\sigma(\omega) \\
&\qquad -(\phi(\omega)M_{ik}(\omega) + \gamma(\omega)L_{ik}(\omega))\beta(\omega)] && (3.36)
\end{aligned}$$

$$\begin{aligned}
U_{ik}(\omega) &= \Im\{[g(j\omega)h(j\omega)]\alpha(j\omega)d_{ik}(j\omega)\} && (3.37)\\
&= \Im\{[\phi(\omega) + j\gamma(\omega)][\sigma(\omega) + j\beta(\omega)][L_{ik}(\omega) + jM_{ik}(\omega)]\} && (3.38)\\
&= [(\phi(\omega)M_{ik}(\omega) + \gamma(\omega)L_{ik}(\omega))\sigma(\omega) \\
&\qquad +(\phi(\omega)L_{ik}(\omega) - \gamma(\omega)M_{ik}(\omega))\beta(\omega)] && (3.39)
\end{aligned}$$

$$X_{ik}(\omega) = \Re\{n_{ik}(j\omega)\} = \Re\left\{\sum_{r=1}^{n} l_{ir}(j\omega)m_{rk}(j\omega)\right\} \tag{3.40}$$

$$= \Re\left\{\sum_{r=1}^{n}[(Q_{ir}(\omega) + jR_{ir}(\omega))(\psi_{rk}(\omega) + j\theta_{rk}(\omega))]\right\} \tag{3.41}$$

$$= \sum_{r=1}^{n}[Q_{ir}(\omega)\psi_{rk}(\omega) - R_{ir}(\omega)\theta_{rk}(\omega)] \tag{3.42}$$

$$= \sum_{r=1}^{n}\sum_{r'=1}^{n}[(A_{ir'}(\omega)\psi_{rk}(\omega) - B_{ir'}(\omega)\theta_{rk}(\omega))\sigma_{r'r}(\omega)$$

$$-(A_{ir'}(\omega)\theta_{rk}(\omega) + B_{ir'}(\omega)\psi_{rk}(\omega))\beta_{r'r}(\omega)] \tag{3.43}$$

$$Y_{ik}(\omega) = \Im\{n_{ik}(j\omega)\} = \Im\left\{\sum_{r=1}^{n} l_{ir}(j\omega)m_{rk}(j\omega)\right\} \tag{3.44}$$

$$= \Im\left\{\sum_{r=1}^{n}[(Q_{ir}(\omega) + jR_{ir}(\omega))(\psi_{rk}(\omega) + j\theta_{rk}(\omega))]\right\} \tag{3.45}$$

$$= \sum_{r=1}^{n}[Q_{ir}(\omega)\theta_{rk}(\omega) + R_{ir}(\omega)\psi_{rk}(\omega)] \tag{3.46}$$

$$= \sum_{r=1}^{n}\sum_{r'=1}^{n}[(A_{ir'}(\omega)\theta_{rk}(\omega) + B_{ir'}(\omega)\psi_{rk}(\omega))\sigma_{r'r}(\omega)$$

$$+(A_{ir'}(\omega)\psi_{rk}(\omega) - B_{ir'}(\omega)\theta_{rk}(\omega))\beta_{r'r}(\omega)] \tag{3.47}$$

Substituting Equations (3.26), (3.30), (3.34), (3.37), (3.40) and (3.44) into Equation (3.11), we obtain

$$E_m = \sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\omega_{ik}}^{\omega'_{ik}} |\{\mu_{ik}(\omega) + j\lambda_{ik}(\omega)\}$$

$$- \{(T_{ik}(\omega) + jU_{ik}(\omega)) - (X_{ik}(\omega) + jY_{ik}(\omega))\}|^2\, d\omega \tag{3.48}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\omega_{ik}}^{\omega'_{ik}} \Big\{(\mu_{ik}(\omega) - X_{ik}(\omega) - T_{ik}(\omega))^2$$

$$+ (\lambda_{ik}(\omega) - Y_{ik}(\omega) - U_{ik}(\omega))^2\Big\}\, d\omega \tag{3.49}$$

Now we minimize the error function E_m in Equation (3.49) by equating to zero the partial derivatives of E_m with respect to $\alpha_r^{(i,k)}$ and b_z, for $0 \leq r \leq p(i,k)$, $1 \leq i, k \leq n$, and $1 \leq z \leq q$. It will be shown that this will lead to linear algebraic equations involving those unknown controller parameters. For the sake of brevity, from now on, we will express a function of ω just by its function name (i.e., $f = f(\omega)$). Now we divide the partial derivatives of E_m

with respect to those unknown controller parameters into four major partial derivatives as follows:

$$\frac{\partial E_m}{\partial a_r^{(v,w)}} = \frac{\partial E_m}{\partial \sigma_{vw}} \frac{\partial \sigma_{vw}}{\partial a_r^{(v,w)}} = 0 \ , \quad \begin{array}{l} r = 0,2,4,\ldots,p(v,w) \ , \\ 1 \le v,w \le n \end{array} \tag{3.50}$$

$$\frac{\partial E_m}{\partial a_r^{(v,w)}} = \frac{\partial E_m}{\partial \beta_{vw}} \frac{\partial \beta_{vw}}{\partial a_r^{(v,w)}} = 0 \ , \quad \begin{array}{l} r = 1,3,5,\ldots,p(v,w) \ , \\ 1 \le v,w \le n \end{array} \tag{3.51}$$

$$\frac{\partial E_m}{\partial b_r} = \frac{\partial E_m}{\partial \sigma} \frac{\partial \sigma}{\partial b_r} = 0 \ , \ r = 2,4,6,\ldots,q \tag{3.52}$$

$$\frac{\partial E_m}{\partial b_r} = \frac{\partial E_m}{\partial \beta} \frac{\partial \beta}{\partial b_r} = 0 \ , \ r = 1,3,5,\ldots,q \tag{3.53}$$

In order to do the partial derivatives of E_m as in Equation (3.50), we need the following:

$$\begin{aligned} \frac{\partial E_m}{\partial \sigma_{vw}} = \sum_{i=1}^{n} \sum_{k=1}^{n} \int_{\omega_{ik}}^{\omega'_{ik}} & \left[2(\mu_{ik} - X_{ik} - T_{ik}) \left(\frac{\partial \mu_{ik}}{\partial \sigma_{vw}} - \frac{\partial X_{ik}}{\partial \sigma_{vw}} - \frac{\partial T_{ik}}{\partial \sigma_{vw}} \right) \right. \\ & \left. + \, 2(\lambda_{ik} - Y_{ik} - U_{ik}) \left(\frac{\partial \lambda_{ik}}{\partial \sigma_{vw}} - \frac{\partial Y_{ik}}{\partial \sigma_{vw}} - \frac{\partial U_{ik}}{\partial \sigma_{vw}} \right) \right] d\omega \end{aligned} \tag{3.54}$$

where

$$\left. \begin{array}{l} \dfrac{\partial \mu_{ik}}{\partial \sigma_{vw}} = \psi A_{iv} - \theta B_{iv} \ , \quad \dfrac{\partial X_{ik}}{\partial \sigma_{vw}} = A_{iv}\psi_{wk} - B_{iv}\theta_{wk} \ , \quad \dfrac{\partial T_{ik}}{\partial \sigma_{vw}} = 0 \\ \dfrac{\partial \lambda_{ik}}{\partial \sigma_{vw}} = \theta A_{iv} + \psi B_{iv} \ , \quad \dfrac{\partial Y_{ik}}{\partial \sigma_{vw}} = A_{iv}\theta_{wk} + B_{iv}\psi_{wk} \ , \quad \dfrac{\partial U_{ik}}{\partial \sigma_{vw}} = 0 \end{array} \right\} \tag{3.55}$$

Now, let

$$\hat{A}_{ik}^{vw} = \left(\frac{\partial \mu_{ik}}{\partial \sigma_{vw}} - \frac{\partial X_{ik}}{\partial \sigma_{vw}} \right) \tag{3.56}$$

$$= \psi A_{iv} - \theta B_{iv} - A_{iv}\psi_{wk} + B_{iv}\theta_{wk} \tag{3.57}$$

$$\hat{B}_{ik}^{vw} = \left(\frac{\partial \lambda_{ik}}{\partial \sigma_{vw}} - \frac{\partial Y_{ik}}{\partial \sigma_{vw}} \right) \tag{3.58}$$

$$= \theta A_{iv} + \psi B_{iv} - A_{iv}\theta_{wk} - B_{iv}\psi_{wk} \tag{3.59}$$

So that,

$$(\mu_{ik} - X_{ik} - T_{ik})\left(\frac{\partial \mu_{ik}}{\partial \sigma_{vw}} - \frac{\partial X_{ik}}{\partial \sigma_{vw}}\right) =$$

$$\sum_{r=1}^{n}[\hat{A}_{ik}^{vw}(\psi A_{ir} - \theta B_{ir})\sigma_{rk} - \hat{A}_{ik}^{vw}(\theta A_{ir} + \psi B_{ir})\beta_{rk}]$$

$$-\sum_{r=1}^{n}\sum_{r'=1}^{n}[\hat{A}_{ik}^{vw}(A_{ir'}\psi_{rk} - B_{ir'}\theta_{rk})\sigma_{r'r} - \hat{A}_{ik}^{vw}(A_{ir'}\theta_{rk} + B_{ir'}\psi_{rk})\beta_{r'r}]$$

$$-[\hat{A}_{ik}^{vw}(\phi L_{ik} - \gamma M_{ik})\sigma - \hat{A}_{ik}^{vw}(\phi M_{ik} + \gamma L_{ik})\beta] \tag{3.60}$$

$$(\lambda_{ik} - Y_{ik} - U_{ik})\left(\frac{\partial \lambda_{ik}}{\partial \sigma_{vw}} - \frac{\partial Y_{ik}}{\partial \sigma_{vw}}\right) =$$

$$\sum_{r=1}^{n}[\hat{B}_{ik}^{vw}(\theta A_{ir} + \psi B_{ir})\sigma_{rk} + \hat{B}_{ik}^{vw}(\psi A_{ir} - \theta B_{ir})\beta_{rk}]$$

$$-\sum_{r=1}^{n}\sum_{r'=1}^{n}[\hat{B}_{ik}^{vw}(A_{ir'}\theta_{rk} + B_{ir'}\psi_{rk})\sigma_{r'r} + \hat{B}_{ik}^{vw}(A_{ir'}\psi_{rk} - B_{ir'}\theta_{rk})\beta_{r'r}]$$

$$-[\hat{B}_{ik}^{vw}(\phi M_{ik} + \gamma L_{ik})\sigma + \hat{B}_{ik}^{vw}(\phi L_{ik} - \gamma M_{ik})\beta] \tag{3.61}$$

Then

$$(\mu_{ik} - X_{ik} - T_{ik})\left(\frac{\partial \mu_{ik}}{\partial \sigma_{vw}} - \frac{\partial X_{ik}}{\partial \sigma_{vw}}\right) + (\lambda_{ik} - Y_{ik} - U_{ik})\left(\frac{\partial \lambda_{ik}}{\partial \sigma_{vw}} - \frac{\partial Y_{ik}}{\partial \sigma_{vw}}\right)$$

$$= \sum_{r=1}^{n}[(\psi \hat{A}_{ik}^{vw} + \theta \hat{B}_{ik}^{vw})A_{ir} - (\theta \hat{A}_{ik}^{vw} - \psi \hat{B}_{ik}^{vw})B_{ir}]\sigma_{rk}$$

$$-\sum_{r=1}^{n}[(\theta \hat{A}_{ik}^{vw} - \psi \hat{B}_{ik}^{vw})A_{ir} + (\psi \hat{A}_{ik}^{vw} + \theta \hat{B}_{ik}^{vw})B_{ir}]\beta_{rk}$$

$$-\sum_{r=1}^{n}\sum_{r'=1}^{n}[(\hat{A}_{ik}^{vw}\psi_{rk} + \hat{B}_{ik}^{vw}\theta_{rk})A_{ir'} - (\hat{A}_{ik}^{vw}\theta_{rk} - \hat{B}_{ik}^{vw}\psi_{rk})B_{ir'}]\sigma_{r'r}$$

$$+\sum_{r=1}^{n}\sum_{r'=1}^{n}[(\hat{A}_{ik}^{vw}\theta_{rk} - \hat{B}_{ik}^{vw}\psi_{rk})A_{ir'} + (\hat{A}_{ik}^{vw}\psi_{rk} + \hat{B}_{ik}^{vw}\theta_{rk})B_{ir'}]\beta_{r'r}$$

$$-[\hat{A}_{ik}^{vw}(\phi L_{ik} - \gamma M_{ik}) + \hat{B}_{ik}^{vw}(\phi M_{ik} + \gamma L_{ik})]\sigma$$

$$-[\hat{B}_{ik}^{vw}(\phi L_{ik} - \gamma M_{ik}) - \hat{A}_{ik}^{vw}(\phi M_{ik} + \gamma L_{ik})]\beta \tag{3.62}$$

$$= \sum_{r=1}^{n} \hat{I}_{ikr}^{vw}\sigma_{rk} - \sum_{r=1}^{n} \hat{K}_{ikr}^{vw}\beta_{rk} - \sum_{r=1}^{n}\sum_{r'=1}^{n} \hat{J}_{ikrr'}^{vw}\sigma_{r'r} + \sum_{r=1}^{n}\sum_{r'=1}^{n} \hat{L}_{ikrr'}^{vw}\beta_{r'r} - \hat{M}_{ik}^{vw}\sigma - \hat{N}_{ik}^{vw}\beta \tag{3.63}$$

where

$$\left.\begin{aligned}
\hat{I}_{ikr}^{vw} &= \hat{C}_{ik}^{vw} A_{ir} - \hat{D}_{ik}^{vw} B_{ir} \\
\hat{J}_{ikrr'}^{vw} &= \hat{W}_{ikr}^{vw} A_{ir'} - \hat{V}_{ikr}^{vw} B_{ir'} \\
\hat{K}_{ikr}^{vw} &= \hat{D}_{ik}^{vw} A_{ir} + \hat{C}_{ik}^{vw} B_{ir} \\
\hat{L}_{ikrr'}^{vw} &= \hat{V}_{ikr}^{vw} A_{ir'} + \hat{W}_{ikr}^{vw} B_{ir'} \\
\hat{M}_{ik}^{vw} &= \hat{A}_{ik}^{vw}\overline{A}_{ik} + \hat{B}_{ik}^{vw}\overline{B}_{ik} \\
\hat{N}_{ik}^{vw} &= \hat{B}_{ik}^{vw}\overline{A}_{ik} - \hat{A}_{ik}^{vw}\overline{B}_{ik}
\end{aligned}\right\} \tag{3.64}$$

and

$$\left.\begin{aligned}
\overline{A}_{ik} &= \phi L_{ik} - \gamma M_{ik} \, , & \overline{B}_{ik} &= \phi M_{ik} + \gamma L_{ik} \\
\hat{C}_{ik}^{vw} &= \psi \hat{A}_{ik}^{vw} + \theta \hat{B}_{ik}^{vw} \, , & \hat{D}_{ik}^{vw} &= \theta \hat{A}_{ik}^{vw} - \psi \hat{B}_{ik}^{vw} \\
\hat{W}_{ikr}^{vw} &= \hat{A}_{ik}^{vw}\psi_{rk} + \hat{B}_{ik}^{vw}\theta_{rk} \, , & \hat{V}_{ikr}^{vw} &= \hat{A}_{ik}^{vw}\theta_{rk} - \hat{B}_{ik}^{vw}\psi_{rk}
\end{aligned}\right\} \tag{3.65}$$

Now, substituting Equation (3.63) into Equation (3.54), we obtain

$$\begin{aligned}
\frac{\partial E_m}{\partial \sigma_{vw}} = \sum_{i=1}^{n}\sum_{k=1}^{n} \int_{\omega_{ik}}^{\omega'_{ik}} 2 \Bigg[& \sum_{r=1}^{n} \hat{I}_{ikr}^{vw}\sigma_{rk} - \sum_{r=1}^{n}\sum_{r'=1}^{n} \hat{J}_{ikrr'}^{vw}\sigma_{r'r} \\
& - \sum_{r=1}^{n} \hat{K}_{ikr}^{vw}\beta_{rk} + \sum_{r=1}^{n}\sum_{r'=1}^{n} \hat{L}_{ikrr'}^{vw}\beta_{r'r} \\
& - \hat{M}_{ik}^{vw}\sigma - \hat{N}_{ik}^{vw}\beta \Bigg] d\omega
\end{aligned} \tag{3.66}$$

$$
\begin{aligned}
= & \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{n}\int_{\omega_{ik}}^{\omega'_{ik}} 2\hat{I}_{ikr}^{vw}\sigma_{rk}d\omega - \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{n}\sum_{r'=1}^{n}\int_{\omega_{ik}}^{\omega'_{ik}} 2\hat{J}_{ikrr'}^{vw}\sigma_{r'r}d\omega \\
& -\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{n}\int_{\omega_{ik}}^{\omega'_{ik}} 2\hat{K}_{ikr}^{vw}\beta_{rk}d\omega + \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{n}\sum_{r'=1}^{n}\int_{\omega_{ik}}^{\omega'_{ik}} 2\hat{L}_{ikrr'}^{vw}\beta_{r'r}d\omega \\
& -\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\omega_{ik}}^{\omega'_{ik}} 2\hat{M}_{ik}^{vw}\sigma d\omega - \sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\omega_{ik}}^{\omega'_{ik}} 2\hat{N}_{ik}^{vw}\beta d\omega \\
= & \sum_{i=1}^{n}\sum_{k=1}^{n} 2\left\{\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \hat{I}_{rki}^{vw}\sigma_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \hat{J}_{rr'ki}^{vw}\sigma_{ik}d\omega\right]\right. \\
& \qquad -\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \hat{K}_{rki}^{vw}\beta_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \hat{L}_{rr'ki}^{vw}\beta_{ik}d\omega\right]
\end{aligned}
\tag{3.67}
$$

$$
\left. - \left[\int_{\omega_{ik}}^{\omega'_{ik}} \hat{M}_{ik}^{vw}\sigma d\omega\right] - \left[\int_{\omega_{ik}}^{\omega'_{ik}} \hat{N}_{ik}^{vw}\beta d\omega\right]\right\}
\tag{3.68}
$$

since

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{n}\int_{\omega_{ik}}^{\omega'_{ik}} \hat{I}_{ikr}^{vw}\sigma_{rk}d\omega &= \sum_{r=1}^{n}\sum_{k=1}^{n}\sum_{i=1}^{n}\int_{\omega_{ik}}^{\omega'_{ik}} \hat{I}_{ikr}^{vw}\sigma_{rk}d\omega \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{n}\int_{\omega_{rk}}^{\omega'_{rk}} \hat{I}_{rki}^{vw}\sigma_{ik}d\omega \\
& \quad \text{by letting } r = i \text{ and } i = r
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{n}\sum_{r'=1}^{n}\int_{\omega_{ik}}^{\omega'_{ik}} \hat{J}_{ikrr'}^{vw}\sigma_{r'r}d\omega &= \sum_{r'=1}^{n}\sum_{r=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\omega_{ik}}^{\omega'_{ik}} \hat{J}_{ikrr'}^{vw}\sigma_{r'r}d\omega \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{n}\sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \hat{J}_{rr'ik}^{vw}\sigma_{ik}d\omega \\
& \quad \text{by letting } r' = i,\ r = k,\ i = r \text{ and } k = r'
\end{aligned}
$$

Finally, putting it all together as in Equation (3.50), which is

$$
\frac{\partial E_m}{\partial a_z^{(v,w)}} = \frac{\partial E_m}{\partial \sigma_{vw}}\frac{\partial \sigma_{vw}}{\partial a_z^{(v,w)}} = 0\ , \quad \begin{array}{l} z = 0,2,4,\ldots,p(v,w)\ , \\ 1 \le v,w \le n \end{array}
\tag{3.69}
$$

Therefore, we obtain

$$\sum_{i=1}^{n}\sum_{k=1}^{n} 2\left\{\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \frac{\partial\sigma_{vw}}{\partial a_z^{(v,w)}} \hat{I}_{rki}^{vw}\sigma_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \frac{\partial\sigma_{vw}}{\partial a_z^{(v,w)}} \hat{J}_{rr'ki}^{vw}\sigma_{ik}d\omega\right]\right.$$

$$-\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \frac{\partial\sigma_{vw}}{\partial a_z^{(v,w)}} \hat{K}_{rki}^{vw}\beta_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \frac{\partial\sigma_{vw}}{\partial a_z^{(v,w)}} \hat{L}_{rr'ki}^{vw}\beta_{ik}d\omega\right]$$

$$\left.-\left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\sigma_{vw}}{\partial a_z^{(v,w)}} \hat{M}_{ik}^{vw}\sigma d\omega\right] - \left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\sigma_{vw}}{\partial a_z^{(v,w)}} \hat{N}_{ik}^{vw}\beta d\omega\right]\right\}$$

$$= 0 \ , \quad \begin{array}{l} z = 0, 2, 4, \ldots, p(v,w) \ , \\ 1 \le v, w \le n \end{array} \tag{3.70}$$

which implies

$$\sum_{i=1}^{n}\sum_{k=1}^{n} \left\{\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \frac{\partial\sigma_{vw}}{\partial a_z^{(v,w)}} \hat{I}_{rki}^{vw}\sigma_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \frac{\partial\sigma_{vw}}{\partial a_z^{(v,w)}} \hat{J}_{rr'ki}^{vw}\sigma_{ik}d\omega\right]\right.$$

$$-\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \frac{\partial\sigma_{vw}}{\partial a_z^{(v,w)}} \hat{K}_{rki}^{vw}\beta_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \frac{\partial\sigma_{vw}}{\partial a_z^{(v,w)}} \hat{L}_{rr'ki}^{vw}\beta_{ik}d\omega\right]$$

$$\left.-\left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\sigma_{vw}}{\partial a_z^{(v,w)}} \hat{M}_{ik}^{vw}\hat{\sigma} d\omega\right] - \left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\sigma_{vw}}{\partial a_z^{(v,w)}} \hat{N}_{ik}^{vw}\beta d\omega\right]\right\}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n}\left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\sigma_{vw}}{\partial a_z^{(v,w)}} \hat{M}_{ik}^{vw} d\omega\right] \ , \quad \begin{array}{l} z = 0, 2, 4, \ldots, p(v,w) \ , \\ 1 \le v, w \le n \end{array} \tag{3.71}$$

where

$$\hat{\sigma} = \sigma - 1 = -b_2\omega^2 + b_4\omega^4 - b_6\omega^6 + \cdots \tag{3.72}$$

As an example, take z = 0 and fix a value of (v,w). Note that

$$\frac{\partial\sigma_{vw}}{\partial a_0^{(v,w)}} = \frac{\partial}{\partial a_0^{(v,w)}}[a_0^{(v,w)} - a_2^{(v,w)}\omega^2 + \cdots] = 1 \tag{3.73}$$

After some algebraic manipulation Equation (3.71) leads to

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\left\{\sum_{r=1}^{n}\left[\left(\int_{\omega_{rk}}^{\omega'_{rk}} \hat{I}_{rki}^{vw} - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \hat{J}_{rr'ki}^{vw}\right)\sigma_{ik}d\omega\right]\right.$$
$$-\sum_{r=1}^{n}\left[\left(\int_{\omega_{rk}}^{\omega'_{rk}} \hat{K}_{rki}^{vw} - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \hat{L}_{rr'ki}^{vw}\right)\beta_{ik}d\omega\right]$$
$$\left.-\left[\left(\int_{\omega_{ik}}^{\omega'_{ik}} \hat{M}_{ik}^{vw}\right)\hat{\sigma}d\omega\right] - \left[\left(\int_{\omega_{ik}}^{\omega'_{ik}} \hat{N}_{ik}^{vw}\right)\beta d\omega\right]\right\}$$
$$= \sum_{i=1}^{n}\sum_{k=1}^{n}\left[\int_{\omega_{ik}}^{\omega'_{ik}} \hat{M}_{ik}^{vw} d\omega\right] \tag{3.74}$$

We now use the following set of notations

$$\left.\begin{aligned}
T_h^{(i,k,v,w)} &= \sum_{r=1}^{n}\left[\left(\int_{\omega_{rk}}^{\omega'_{rk}} \hat{I}_{rki}^{vw} - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \hat{J}_{rr'ki}^{vw}\right)\omega^h d\omega\right] \\
U_h^{(i,k,v,w)} &= \sum_{r=1}^{n}\left[\left(\int_{\omega_{rk}}^{\omega'_{rk}} \hat{K}_{rki}^{vw} - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \hat{L}_{rr'ki}^{vw}\right)\omega^h d\omega\right] \\
S_h^{(v,w)} &= \sum_{i=1}^{n}\sum_{k=1}^{n}\left[\int_{\omega_{ik}}^{\omega'_{ik}} \hat{M}_{ik}^{vw}\omega^h d\omega\right] \\
R_h^{(v,w)} &= \sum_{i=1}^{n}\sum_{k=1}^{n}\left[\int_{\omega_{ik}}^{\omega'_{ik}} \hat{N}_{ik}^{vw}\omega^h d\omega\right]
\end{aligned}\right\} \tag{3.75}$$

Since σ_{ik}, β_{ik}, σ and β are given as in Equation (3.16) and using the above notations, Equation (3.75), we can rewrite Equation (3.74) as below:

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\left[T_0^{(i,k,v,w)}a_0^{(i,k)} - T_2^{(i,k,v,w)}a_2^{(i,k)} + \cdots\right]$$
$$-\sum_{i=1}^{n}\sum_{k=1}^{n}\left[U_1^{(i,k,v,w)}a_1^{(i,k)} - U_3^{(i,k,v,w)}a_3^{(i,k)} + \cdots\right]$$
$$-\left[-S_2^{(v,w)}b_2 + S_4^{(v,w)}b_4 - \cdots\right]$$
$$-\left[R_1^{(v,w)}b_1 - R_3^{(v,w)}b_3 + \cdots\right] = S_0^{(v,w)} \tag{3.76}$$

We can similarly generate equations similar to Equation (3.76) by taking partial derivatives of E_m with respect to $a_z^{(v,\omega)}$ for z = 2,4,6...,p(v,ω) and $1 \leq v, \omega \leq n$. These equations can be written in a matrix form,

$$\mathbf{X}_1\mathbf{Y}_1 = \mathbf{Z}_1 \tag{3.77}$$

where the matrix $\mathbf{X}_1$ is given by Table 3.1

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ \cdots & T_0^{(i,k,v,w)} & -U_1^{(i,k,v,w)} & -T_2^{(i,k,v,w)} & \cdots & -R_1^{(v,w)} & S_2^{(v,w)} & R_3^{(v,w)} & \cdots \\ \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots \\ \cdots & -T_2^{(i,k,v,w)} & U_3^{(i,k,v,w)} & T_4^{(i,k,v,w)} & \cdots & R_3^{(v,w)} & -S_4^{(v,w)} & -R_5^{(v,w)} & \cdots \\ \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots \\ \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{3.78}$$

Table 3.1: The matrix $\mathbf{X}_1$

and

$$\mathbf{Y}_1 = \begin{bmatrix} \vdots \\ a_0^{(i,k)} \\ a_1^{(i,k)} \\ a_2^{(i,k)} \\ \vdots \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}, \quad \mathbf{Z}_1 = \begin{bmatrix} \vdots \\ S_0^{(v,w)} \\ \cdot \\ -S_2^{(v,w)} \\ \cdot \\ \vdots \end{bmatrix} \tag{3.79}$$

Now we establish similar relationships as above by doing the partial derivatives of E_m as in Equation (3.51). First, note that

$$\begin{aligned} \frac{\partial E_m}{\partial \beta_{vw}} = & \sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\omega_{ik}}^{\omega'_{ik}} \left[2(\mu_{ik} - X_{ik} - T_{ik})\left(\frac{\partial \mu_{ik}}{\partial \beta_{vw}} - \frac{\partial X_{ik}}{\partial \beta_{vw}} - \frac{\partial T_{ik}}{\partial \beta_{vw}}\right)\right. \\ & \left. +2\left(\lambda_{ik} - Y_{ik} - U_{ik}\right)\left(\frac{\partial \lambda_{ik}}{\partial \beta_{vw}} - \frac{\partial Y_{ik}}{\partial \beta_{vw}} - \frac{\partial U_{ik}}{\partial \beta_{vw}}\right)\right] d\omega \end{aligned} \tag{3.80}$$

where

$$
\left.\begin{aligned}
\frac{\partial \mu_{ik}}{\partial \beta_{vw}} &= -(\theta A_{iv} + \psi B_{iv}) \;, \quad \frac{\partial X_{ik}}{\partial \beta_{vw}} = -(A_{iv}\theta_{wk} + B_{iv}\psi_{wk}) \;, \quad \frac{\partial T_{ik}}{\partial \beta_{vw}} = 0 \\
\frac{\partial \lambda_{ik}}{\partial \beta_{vw}} &= \psi A_{iv} - \theta B_{iv} \;, \qquad \frac{\partial Y_{ik}}{\partial \beta_{vw}} = A_{iv}\psi_{wk} - B_{iv}\theta_{wk} \;, \qquad \frac{\partial U_{ik}}{\partial \beta_{vw}} = 0
\end{aligned}\right\} \tag{3.81}
$$

Now,

$$
\left(\frac{\partial \mu_{ik}}{\partial \beta_{vw}} - \frac{\partial X_{ik}}{\partial \beta_{vw}}\right) = -(\theta A_{iv} + \psi B_{iv} - A_{iv}\theta_{wk} - B_{iv}\psi_{wk}) = -\hat{B}_{ik}^{vw} \tag{3.82}
$$

$$
\left(\frac{\partial \lambda_{ik}}{\partial \beta_{vw}} - \frac{\partial Y_{ik}}{\partial \beta_{vw}}\right) = \psi A_{iv} - \theta B_{iv} - A_{iv}\psi_{wk} + B_{iv}\theta_{wk} = \hat{A}_{ik}^{vw} \tag{3.83}
$$

Thus, we have

$$
\begin{aligned}
&(\mu_{ik} - X_{ik} - T_{ik})\left(\frac{\partial \mu_{ik}}{\partial \beta_{vw}} - \frac{\partial X_{ik}}{\partial \beta_{vw}}\right) = \\
&\quad \sum_{r=1}^{n}[-\hat{B}_{ik}^{vw}(\psi A_{ir} - \theta B_{ir})\sigma_{rk} + \hat{B}_{ik}^{vw}(\theta A_{ir} + \psi B_{ir})\beta_{rk}] \\
&\quad + \sum_{r=1}^{n}\sum_{r'=1}^{n}[\hat{B}_{ik}^{vw}(A_{ir'}\psi_{rk} - B_{ir'}\theta_{rk})\sigma_{r'r} - \hat{B}_{ik}^{vw}(A_{ir'}\theta_{rk} + B_{ir'}\psi_{rk})\beta_{r'r}] \\
&\quad + [\hat{B}_{ik}^{vw}(\phi L_{ik} - \gamma M_{ik})\sigma - \hat{B}_{ik}^{vw}(\phi M_{ik} + \gamma L_{ik})\beta]
\end{aligned} \tag{3.84}
$$

$$
\begin{aligned}
&(\lambda_{ik} - Y_{ik} - U_{ik})\left(\frac{\partial \lambda_{ik}}{\partial \beta_{vw}} - \frac{\partial Y_{ik}}{\partial \beta_{vw}}\right) = \\
&\quad \sum_{r=1}^{n}[\hat{A}_{ik}^{vw}(\theta A_{ir} + \psi B_{ir})\sigma_{rk} + \hat{A}_{ik}^{vw}(\psi A_{ir} - \theta B_{ir})\beta_{rk}] \\
&\quad - \sum_{r=1}^{n}\sum_{r'=1}^{n}[\hat{A}_{ik}^{vw}(A_{ir'}\theta_{rk} + B_{ir'}\psi_{rk})\sigma_{r'r} + \hat{A}_{ik}^{vw}(A_{ir'}\psi_{rk} - B_{ir'}\theta_{rk})\beta_{r'r}] \\
&\quad - [\hat{A}_{ik}^{vw}(\phi M_{ik} + \gamma L_{ik})\sigma + \hat{A}_{ik}^{vw}(\phi L_{ik} - \gamma M_{ik})\beta]
\end{aligned} \tag{3.85}
$$

Then

$$
\begin{aligned}
&(\mu_{ik} - X_{ik} - T_{ik})\left(\frac{\partial \mu_{ik}}{\partial \beta_{vw}} - \frac{\partial X_{ik}}{\partial \beta_{vw}}\right) + (\lambda_{ik} - Y_{ik} - U_{ik})\left(\frac{\partial \lambda_{ik}}{\partial \beta_{vw}} - \frac{\partial Y_{ik}}{\partial \beta_{vw}}\right) \\
&= \sum_{r=1}^{n}[(\theta \hat{A}_{ik}^{vw} - \psi \hat{B}_{ik}^{vw})A_{ir} + (\psi \hat{A}_{ik}^{vw} + \theta \hat{B}_{ik}^{vw})B_{ir}]\sigma_{rk} \\
&\quad + \sum_{r=1}^{n}[(\psi \hat{A}_{ik}^{vw} + \theta \hat{B}_{ik}^{vw})A_{ir} - (\theta \hat{A}_{ik}^{vw} - \psi \hat{B}_{ik}^{vw})B_{ir}]\beta_{rk} \\
&\quad - \sum_{r=1}^{n}\sum_{r'=1}^{n}[(\hat{A}_{ik}^{vw}\theta_{rk} - \hat{B}_{ik}^{vw}\psi_{rk})A_{ir'} + (\hat{A}_{ik}^{vw}\psi_{rk} + \hat{B}_{ik}^{vw}\theta_{rk})B_{ir'}]\sigma_{r'r} \\
&\quad - \sum_{r=1}^{n}\sum_{r'=1}^{n}[(\hat{A}_{ik}^{vw}\psi_{rk} + \hat{B}_{ik}^{vw}\theta_{rk})A_{ir'} - (\hat{A}_{ik}^{vw}\theta_{rk} - \hat{B}_{ik}^{vw}\psi_{rk})B_{ir'}]\beta_{r'r} \\
&\quad + [\hat{B}_{ik}^{vw}(\phi L_{ik} - \gamma M_{ik}) - \hat{A}_{ik}^{vw}(\phi M_{ik} + \gamma L_{ik})]\sigma \\
&\quad - [\hat{A}_{ik}^{vw}(\phi L_{ik} - \gamma M_{ik}) + \hat{B}_{ik}^{vw}(\phi M_{ik} + \gamma L_{ik})]\beta
\end{aligned} \tag{3.86}
$$

$$
\begin{aligned}
&= \sum_{r=1}^{n}\hat{K}_{ikr}^{vw}\sigma_{rk} + \sum_{r=1}^{n}\hat{I}_{ikr}^{vw}\beta_{rk} - \sum_{r=1}^{n}\sum_{r'=1}^{n}\hat{L}_{ikrr'}^{vw}\sigma_{r'r} - \sum_{r=1}^{n}\sum_{r'=1}^{n}\hat{J}_{ikrr'}^{vw}\beta_{r'r} \\
&\quad + \hat{N}_{ik}^{vw}\sigma - \hat{M}_{ik}^{vw}\beta
\end{aligned} \tag{3.87}
$$

by using the same notation as in Equations (3.64) and (3.65).

As before, by substituting Equation (3.87) into Equation (3.80) and by using the argument similar to the one used earlier, we obtain

$$
\begin{aligned}
\frac{\partial E_m}{\partial \beta_{vw}} = \sum_{i=1}^{n}\sum_{k=1}^{n} 2\Bigg\{ &\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \hat{K}_{rki}^{vw}\sigma_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \hat{L}_{rr'ki}^{vw}\sigma_{ik}d\omega\right] \\
&+ \sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \hat{I}_{rki}^{vw}\beta_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \hat{J}_{rr'ki}^{vw}\beta_{ik}d\omega\right] \\
&+ \left[\int_{\omega_{ik}}^{\omega'_{ik}} \hat{N}_{ik}^{vw}\sigma d\omega\right] - \left[\int_{\omega_{ik}}^{\omega'_{ik}} \hat{M}_{ik}^{vw}\beta d\omega\right]\Bigg\}
\end{aligned} \tag{3.88}
$$

Finally, putting it all together as in Equation (3.51), which is

$$
\frac{\partial E_m}{\partial a_z^{(v,w)}} = \frac{\partial E_m}{\partial \beta_{vw}}\frac{\partial \beta_{vw}}{\partial a_z^{(v,w)}} = 0\,, \quad \begin{array}{l} z = 1,3,5,\ldots,p(v,w)\,, \\ 1 \le v,w \le n \end{array} \tag{3.89}
$$

Therefore, we obtain

$$\sum_{i=1}^{n}\sum_{k=1}^{n} 2\left\{\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \frac{\partial\beta_{vw}}{\partial a_z^{(v,w)}}\hat{K}_{rki}^{vw}\sigma_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \frac{\partial\beta_{vw}}{\partial a_z^{(v,w)}}\hat{L}_{rr'ki}^{vw}\sigma_{ik}d\omega\right]\right.$$
$$+\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \frac{\partial\beta_{vw}}{\partial a_z^{(v,w)}}\hat{I}_{rki}^{vw}\beta_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \frac{\partial\beta_{vw}}{\partial a_z^{(v,w)}}\hat{J}_{rr'ki}^{vw}\beta_{ik}d\omega\right]$$
$$\left.+\left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\beta_{vw}}{\partial a_z^{(v,w)}}\hat{N}_{ik}^{vw}\sigma d\omega\right] - \left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\beta_{vw}}{\partial a_z^{(v,w)}}\hat{M}_{ik}^{vw}\beta d\omega\right]\right\}$$
$$z = 1,3,5,\ldots,p(v,w)\,,$$
$$= 0\,, \qquad 1 \le v,w \le n \tag{3.90}$$

which implies

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\left\{\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \frac{\partial\beta_{vw}}{\partial a_z^{(v,w)}}\hat{K}_{rki}^{vw}\sigma_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \frac{\partial\beta_{vw}}{\partial a_z^{(v,w)}}\hat{L}_{rr'ki}^{vw}\sigma_{ik}d\omega\right]\right.$$
$$+\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \frac{\partial\beta_{vw}}{\partial a_z^{(v,w)}}\hat{I}_{rki}^{vw}\beta_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \frac{\partial\beta_{vw}}{\partial a_z^{(v,w)}}\hat{J}_{rr'ki}^{vw}\beta_{ik}d\omega\right]$$
$$\left.+\left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\beta_{vw}}{\partial a_z^{(v,w)}}\hat{N}_{ik}^{vw}\hat{\sigma} d\omega\right] - \left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\beta_{vw}}{\partial a_z^{(v,w)}}\hat{M}_{ik}^{vw}\beta d\omega\right]\right\}$$
$$= -\sum_{i=1}^{n}\sum_{k=1}^{n}\left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\beta_{vw}}{\partial a_z^{(v,w)}}\hat{N}_{ik}^{vw}d\omega\right]\,, \qquad z = 1,3,5,\ldots,p(v,w)\,,\quad 1 \le v,w \le n \tag{3.91}$$

where $\hat{\sigma}$ is given by Equation (3.72).

As an example, take z = 1 and fix a value of (v,ω). Note that

$$\frac{\partial\beta_{vw}}{\partial a_1^{(v,w)}} = \frac{\partial}{\partial a_1^{(v,w)}}[a_1^{(v,w)}\omega - a_3^{(v,w)}\omega^3 + \cdots] = \omega \tag{3.92}$$

After some algebraic manipulation Equation (3.91) leads to

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\left\{\sum_{r=1}^{n}\left[\left(\int_{\omega_{rk}}^{\omega'_{rk}}\omega\hat{K}_{rki}^{vw} - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}}\omega\hat{L}_{rr'ki}^{vw}\right)\sigma_{ik}d\omega\right]\right.$$
$$+\sum_{r=1}^{n}\left[\left(\int_{\omega_{rk}}^{\omega'_{rk}}\omega\hat{I}_{rki}^{vw} - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}}\omega\hat{J}_{rr'ki}^{vw}\right)\beta_{ik}d\omega\right]$$
$$\left.+\left[\left(\int_{\omega_{ik}}^{\omega'_{ik}}\omega\hat{N}_{ik}^{vw}\right)\hat{\sigma}d\omega\right] - \left[\left(\int_{\omega_{ik}}^{\omega'_{ik}}\omega\hat{M}_{ik}^{vw}\right)\beta d\omega\right]\right\}$$
$$= -\sum_{i=1}^{n}\sum_{k=1}^{n}\left[\int_{\omega_{ik}}^{\omega'_{ik}}\omega\hat{N}_{ik}^{vw}d\omega\right] \tag{3.93}$$

By using the same notation as in Equations (3.75) and using σ_{ik}, β_{ik}, σ and β as given by Equation (3.16), we can rewrite Equation (3.93) as follows

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{k=1}^{n}\left[U_1^{(i,k,v,w)}a_0^{(i,k)} - U_3^{(i,k,v,w)}a_2^{(i,k)} + \cdots\right] \\
&+\sum_{i=1}^{n}\sum_{k=1}^{n}\left[T_2^{(i,k,v,w)}a_1^{(i,k)} - T_4^{(i,k,v,w)}a_3^{(i,k)} + \cdots\right] \\
&+\left[-R_3^{(v,w)}b_2 + R_5^{(v,w)}b_4 - \cdots\right] \\
&-\left[S_2^{(v,w)}b_1 - S_4^{(v,w)}b_3 + \cdots\right] = -R_1^{(v,w)}
\end{aligned} \tag{3.94}$$

We can similarly generate equations similar to Equation (3.94) by taking partial derivatives of E_m with respect to $\alpha_z^{(v,\omega)}$ for $z = 3,5,7,\ldots p(v,\omega)$ and $1 \leq v,\ \omega \leq n$. These equations can be written in matrix form,

$$\mathbf{X_2 Y_2 = Z_2} \tag{3.95}$$

where the matrix $\mathbf{X_2}$ is given by Table 3.2.

$$\begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
\cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots \\
\cdots & U_1^{(i,k,v,w)} & T_2^{(i,k,v,w)} & -U_3^{(i,k,v,w)} & \cdots & -S_2^{(v,w)} & -R_3^{(v,w)} & S_4^{(v,w)} & \cdots \\
\cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots \\
\cdots & -U_3^{(i,k,v,w)} & -T_4^{(i,k,v,w)} & U_5^{(i,k,v,w)} & \cdots & S_4^{(v,w)} & R_5^{(v,w)} & -S_6^{(v,w)} & \cdots \\
\ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \tag{3.96}$$

Table 3.2: The matrix $\mathbf{X_2}$

and

$$
\mathbf{Y_2} = \begin{bmatrix} \vdots \\ a_0^{(i,k)} \\ a_1^{(i,k)} \\ a_2^{(i,k)} \\ \vdots \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}, \quad \mathbf{Z_2} = \begin{bmatrix} \vdots \\ . \\ -R_1^{(v,w)} \\ . \\ R_3^{(v,w)} \\ . \\ \vdots \end{bmatrix} \tag{3.97}
$$

Now we establish similar relationships as above by doing the partial derivatives of E_m with respect to b's as in Equation (3.52). First, note that

$$
\begin{aligned}
\frac{\partial E_m}{\partial \sigma} &= \sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\omega_{ik}}^{\omega'_{ik}} \left[2(\mu_{ik} - X_{ik} - T_{ik})\left(\frac{\partial \mu_{ik}}{\partial \sigma} - \frac{\partial X_{ik}}{\partial \sigma} - \frac{\partial T_{ik}}{\partial \sigma}\right)\right. \\
&\quad \left. + 2(\lambda_{ik} - Y_{ik} - U_{ik})\left(\frac{\partial \lambda_{ik}}{\partial \sigma} - \frac{\partial Y_{ik}}{\partial \sigma} - \frac{\partial U_{ik}}{\partial \sigma}\right)\right] d\omega
\end{aligned} \tag{3.98}
$$

where

$$
\left.\begin{aligned}
\frac{\partial \mu_{ik}}{\partial \sigma} = 0\,, \quad \frac{\partial X_{ik}}{\partial \sigma} = 0\,, \quad \frac{\partial T_{ik}}{\partial \sigma} = \phi L_{ik} - \gamma M_{ik} = \overline{A}_{ik} \\
\frac{\partial \lambda_{ik}}{\partial \sigma} = 0\,, \quad \frac{\partial Y_{ik}}{\partial \sigma} = 0\,, \quad \frac{\partial U_{ik}}{\partial \sigma} = \phi M_{ik} + \gamma L_{ik} = \overline{B}_{ik}
\end{aligned}\right\} \tag{3.99}
$$

Thus, we have

$$(\mu_{ik} - X_{ik} - T_{ik})\left(-\frac{\partial T_{ik}}{\partial \sigma}\right) =$$

$$- \sum_{r=1}^{n} [\overline{A}_{ik}(\psi A_{ir} - \theta B_{ir})\sigma_{rk} - \overline{A}_{ik}(\theta A_{ir} + \psi B_{ir})\beta_{rk}]$$
$$+ \sum_{r=1}^{n} \sum_{r'=1}^{n} [\overline{A}_{ik}(A_{ir'}\psi_{rk} - B_{ir'}\theta_{rk})\sigma_{r'r} - \overline{A}_{ik}(A_{ir'}\theta_{rk} + B_{ir'}\psi_{rk})\beta_{r'r}]$$
$$+ [\overline{A}_{ik}(\phi L_{ik} - \gamma M_{ik})\sigma - \overline{A}_{ik}(\phi M_{ik} + \gamma L_{ik})\beta] \tag{3.100}$$

$$(\lambda_{ik} - Y_{ik} - U_{ik})\left(-\frac{\partial U_{ik}}{\partial \sigma}\right) =$$

$$- \sum_{r=1}^{n} [\overline{B}_{ik}(\theta A_{ir} + \psi B_{ir})\sigma_{rk} + \overline{B}_{ik}(\psi A_{ir} - \theta B_{ir})\beta_{rk}]$$
$$+ \sum_{r=1}^{n} \sum_{r'=1}^{n} [\overline{B}_{ik}(A_{ir'}\theta_{rk} + B_{ir'}\psi_{rk})\sigma_{r'r} + \overline{B}_{ik}(A_{ir'}\psi_{rk} - B_{ir'}\theta_{rk})\beta_{r'r}]$$
$$+ [\overline{B}_{ik}(\phi M_{ik} + \gamma L_{ik})\sigma + \overline{B}_{ik}(\phi L_{ik} - \gamma M_{ik})\beta] \tag{3.101}$$

Then

$$(\mu_{ik} - X_{ik} - T_{ik})\left(-\frac{\partial T_{ik}}{\partial \sigma}\right) + (\lambda_{ik} - Y_{ik} - U_{ik})\left(-\frac{\partial U_{ik}}{\partial \sigma}\right)$$

$$\begin{aligned}
= \; & - \sum_{r=1}^{n} [(\psi\overline{A}_{ik} + \theta\overline{B}_{ik})A_{ir} - (\theta\overline{A}_{ik} - \psi\overline{B}_{ik})B_{ir}]\sigma_{rk} \\
& + \sum_{r=1}^{n} [(\theta\overline{A}_{ik} - \psi\overline{B}_{ik})A_{ir} + (\psi\overline{A}_{ik} + \theta\overline{B}_{ik})B_{ir}]\beta_{rk} \\
& + \sum_{r=1}^{n} \sum_{r'=1}^{n} [(\overline{A}_{ik}\psi_{rk} + \overline{B}_{ik}\theta_{rk})A_{ir'} - (\overline{A}_{ik}\theta_{rk} - \overline{B}_{ik}\psi_{rk})B_{ir'}]\sigma_{r'r} \\
& - \sum_{r=1}^{n} \sum_{r'=1}^{n} [(\overline{A}_{ik}\theta_{rk} - \overline{B}_{ik}\psi_{rk})A_{ir'} + (\overline{A}_{ik}\psi_{rk} + \overline{B}_{ik}\theta_{rk})B_{ir'}]\beta_{r'r} \\
& + [\overline{A}_{ik}\overline{A}_{ik} + \overline{B}_{ik}\overline{B}_{ik}]\sigma + [\overline{B}_{ik}\overline{A}_{ik} - \overline{A}_{ik}\overline{B}_{ik}]\beta
\end{aligned} \tag{3.102}$$

$$\begin{aligned}
= \; & - \sum_{r=1}^{n} \overline{I}_{ikr}\sigma_{rk} + \sum_{r=1}^{n} \overline{K}_{ikr}\beta_{rk} + \sum_{r=1}^{n} \sum_{r'=1}^{n} \overline{J}_{ikrr'}\sigma_{r'r} - \sum_{r=1}^{n} \sum_{r'=1}^{n} \overline{L}_{ikrr'}\beta_{r'r} \\
& + \overline{M}_{ik}\sigma + \overline{N}_{ik}\beta
\end{aligned} \tag{3.103}$$

where

$$\left.\begin{aligned}
\overline{I}_{ikr} &= \overline{C}_{ik}A_{ir} - \overline{D}_{ik}B_{ir} \\
\overline{J}_{ikrr'} &= \overline{W}_{ikr}A_{ir'} - \overline{V}_{ikr}B_{ir'} \\
\overline{K}_{ikr} &= \overline{D}_{ik}A_{ir} + \overline{C}_{ik}B_{ir} \\
\overline{L}_{ikrr'} &= \overline{V}_{ikr}A_{ir'} + \overline{W}_{ikr}B_{ir'} \\
\overline{M}_{ik} &= \overline{A}_{ik}\overline{A}_{ik} + \overline{B}_{ik}\overline{B}_{ik} \\
\overline{N}_{ik} &= \overline{B}_{ik}\overline{A}_{ik} - \overline{A}_{ik}\overline{B}_{ik} = 0
\end{aligned}\right\} \tag{3.104}$$

and

$$\left.\begin{aligned}
\overline{A}_{ik} &= \phi L_{ik} - \gamma M_{ik} \ , & \overline{B}_{ik} &= \phi M_{ik} - \gamma L_{ik} \\
\overline{C}_{ik} &= \psi \overline{A}_{ik} + \theta \overline{B}_{ik} \ , & \overline{D}_{ik} &= \theta \overline{A}_{ik} - \psi \overline{B}_{ik} \\
\overline{W}_{ikr} &= \overline{A}_{ik}\psi_{rk} + \overline{B}_{ik}\theta_{rk} \ , & \overline{V}_{ikr} &= \overline{A}_{ik}\theta_{rk} - \overline{B}_{ik}\psi_{rk}
\end{aligned}\right\} \tag{3.105}$$

As before, by substituting Equation (3.103) into Equation (3.98) and by using the argument similar to the one used earlier, we obtain

$$\begin{aligned}
\frac{\partial E_m}{\partial \sigma} = \sum_{i=1}^{n}\sum_{k=1}^{n} 2 \Bigg\{ & -\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \overline{I}_{rki}\sigma_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \overline{J}_{rr'ki}\sigma_{ik}d\omega\right] \\
& + \sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \overline{K}_{rki}\beta_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \overline{L}_{rr'ki}\beta_{ik}d\omega\right] \\
& + \left[\int_{\omega_{ik}}^{\omega'_{ik}} \overline{M}_{ik}\sigma d\omega\right] + \left[\int_{\omega_{ik}}^{\omega'_{ik}} \overline{N}_{ik}\beta d\omega\right]\Bigg\}
\end{aligned} \tag{3.106}$$

Finally, putting it all together as in Equation (3.52), and noting that

$$\frac{\partial E_m}{\partial b_z} = \frac{\partial E_m}{\partial \sigma}\frac{\partial \sigma}{\partial b_z} = 0 \ , \ z = 2, 4, 6, \ldots, q \tag{3.107}$$

We get,

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{k=1}^{n} 2\Bigg\{-\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \frac{\partial\sigma}{\partial b_z}\overline{I}_{rki}\sigma_{ik}\,d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \frac{\partial\sigma}{\partial b_z}\overline{J}_{rr'ki}\sigma_{ik}\,d\omega\right] \\
&\quad + \sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \frac{\partial\sigma}{\partial b_z}\overline{K}_{rki}\beta_{ik}\,d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \frac{\partial\sigma}{\partial b_z}\overline{L}_{rr'ki}\beta_{ik}\,d\omega\right] \\
&\quad + \left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\sigma}{\partial b_z}\overline{M}_{ik}\sigma\,d\omega\right] + \left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\sigma}{\partial b_z}\overline{N}_{ik}\beta\,d\omega\right]\Bigg\} \\
&\quad = 0\ ,\ z = 2,4,6,\ldots,q
\end{aligned} \tag{3.108}
$$

which implies

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{k=1}^{n} \Bigg\{-\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \frac{\partial\sigma}{\partial b_z}\overline{I}_{rki}\sigma_{ik}\,d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \frac{\partial\sigma}{\partial b_z}\overline{J}_{rr'ki}\sigma_{ik}\,d\omega\right] \\
&\quad + \sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \frac{\partial\sigma}{\partial b_z}\overline{K}_{rki}\beta_{ik}\,d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \frac{\partial\sigma}{\partial b_z}\overline{L}_{rr'ki}\beta_{ik}\,d\omega\right] \\
&\quad + \left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\sigma}{\partial b_z}\overline{M}_{ik}\hat{\sigma}\,d\omega\right] + \left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\sigma}{\partial b_z}\overline{N}_{ik}\beta\,d\omega\right]\Bigg\} \\
&\quad = -\sum_{i=1}^{n}\sum_{k=1}^{n}\left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\sigma}{\partial b_z}\overline{M}_{ik}\,d\omega\right]\ ,\ z = 2,4,6,\ldots,q
\end{aligned} \tag{3.109}
$$

where $\hat{\sigma}$ is given as in Equation (3.72).

As an example, take z = 2. Since

$$
\frac{\partial\sigma}{\partial b_2} = \frac{\partial}{\partial b_2}[-b_2\omega^2 + b_4\omega^4 + \cdots] = -\omega^2 \tag{3.110}
$$

It can be easily seen that Equation (3.109) leads to

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{k=1}^{n} \Bigg\{\sum_{r=1}^{n}\left[\left(\int_{\omega_{rk}}^{\omega'_{rk}} \omega^2\overline{I}_{rki} - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \omega^2\overline{J}_{rr'ki}\right)\sigma_{ik}\,d\omega\right] \\
&\quad - \sum_{r=1}^{n}\left[\left(\int_{\omega_{rk}}^{\omega'_{rk}} \omega^2\overline{K}_{rki} - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \omega^2\overline{L}_{rr'ki}\right)\beta_{ik}\,d\omega\right] \\
&\quad - \left[\left(\int_{\omega_{ik}}^{\omega'_{ik}} \omega^2\overline{M}_{ik}\right)\hat{\sigma}\,d\omega\right] - \left[\left(\int_{\omega_{ik}}^{\omega'_{ik}} \omega^2\overline{N}_{ik}\right)\beta\,d\omega\right]\Bigg\} \\
&\quad = \sum_{i=1}^{n}\sum_{k=1}^{n}\left[\int_{\omega_{ik}}^{\omega'_{ik}} \omega^2\overline{M}_{ik}\,d\omega\right]
\end{aligned} \tag{3.111}
$$

We now use the following set of notations:

$$
\left.\begin{aligned}
P_h^{(i,k)} &= \sum_{r=1}^{n}\left[\left(\int_{\omega_{rk}}^{\omega'_{rk}} \overline{I}_{rki} - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \overline{J}_{rr'ki}\right)\omega^h d\omega\right] \\
Q_h^{(i,k)} &= \sum_{r=1}^{n}\left[\left(\int_{\omega_{rk}}^{\omega'_{rk}} \overline{K}_{rki} - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \overline{L}_{rr'ki}\right)\omega^h d\omega\right] \\
Z_h &= \sum_{i=1}^{n}\sum_{k=1}^{n}\left[\int_{\omega_{ik}}^{\omega'_{ik}} \overline{M}_{ik}\omega^h d\omega\right]
\end{aligned}\right\} \tag{3.112}
$$

Using σ_{ik}, β_{ik}, σ and β as given by Equation (3.16), we can rewrite Equation (3.111) as follows:

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{k=1}^{n}\left[P_2^{(i,k)}a_0^{(i,k)} - P_4^{(i,k)}a_2^{(i,k)} + \cdots\right] \\
&-\sum_{i=1}^{n}\sum_{k=1}^{n}\left[Q_3^{(i,k)}a_1^{(i,k)} - Q_5^{(i,k)}a_3^{(i,k)} + \cdots\right] \\
&-\left[-Z_4 b_2 + Z_6 b_4 - \cdots\right] = Z_2
\end{aligned} \tag{3.113}
$$

We can similarly generate equations similar to Equation (3.113) by taking partial derivatives of E_m with respect to b_z for $z = 4,6,8,\ldots,q$. These equations can be written in a matrix form,

$$
\mathbf{X_3 Y_3 = Z_3} \tag{3.114}
$$

where

$$
\mathbf{X_3} = \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
\cdots & . & . & . & . & \cdots & . & . & . & \cdots \\
\cdots & P_2^{(i,k)} & -Q_3^{(i,k)} & -P_4^{(i,k)} & Q_5^{(i,k)} & \cdots & 0 & Z_4 & 0 & \cdots \\
\cdots & . & . & . & . & \cdots & . & . & . & \cdots \\
\cdots & -P_4^{(i,k)} & Q_5^{(i,k)} & P_6^{(i,k)} & -Q_7^{(i,k)} & \cdots & 0 & -Z_6 & 0 & \cdots \\
\cdots & . & . & . & . & \cdots & . & . & . & \cdots \\
\ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \tag{3.115}
$$

Table 3.3: The Matrix $\mathbf{X_3}$.

$$
\mathbf{Y_3} = \begin{bmatrix} \vdots \\ a_0^{(i,k)} \\ a_1^{(i,k)} \\ a_2^{(i,k)} \\ a_3^{(i,k)} \\ \vdots \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}, \quad \mathbf{Z_3} = \begin{bmatrix} \vdots \\ . \\ Z_2 \\ . \\ -Z_4 \\ . \\ \vdots \end{bmatrix} \tag{3.116}
$$

Now we establish similar relationships as above by doing the partial derivatives of $\mathrm{E_m}$ as in Equation (3.53). First,

$$
\begin{aligned}
\frac{\partial E_m}{\partial \beta} = \sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\omega_{ik}}^{\omega'_{ik}} \Bigg[& 2(\mu_{ik} - X_{ik} - T_{ik})\left(\frac{\partial \mu_{ik}}{\partial \beta} - \frac{\partial X_{ik}}{\partial \beta} - \frac{\partial T_{ik}}{\partial \beta}\right) \\
& + 2(\lambda_{ik} - Y_{ik} - U_{ik})\left(\frac{\partial \lambda_{ik}}{\partial \beta} - \frac{\partial Y_{ik}}{\partial \beta} - \frac{\partial U_{ik}}{\partial \beta}\right)\Bigg] d\omega
\end{aligned} \tag{3.117}
$$

where

$$
\left.
\begin{array}{lll}
\frac{\partial \mu_{ik}}{\partial \beta} = 0 \,, & \frac{\partial X_{ik}}{\partial \beta} = 0 \,, & \frac{\partial T_{ik}}{\partial \beta} = -(\phi M_{ik} + \gamma L_{ik}) = -\overline{B}_{ik} \\
\frac{\partial \lambda_{ik}}{\partial \beta} = 0 \,, & \frac{\partial Y_{ik}}{\partial \beta} = 0 \,, & \frac{\partial U_{ik}}{\partial \beta} = \phi L_{ik} - \gamma M_{ik} = \overline{A}_{ik}
\end{array}
\right\} \tag{3.118}
$$

Thus, we have

$$(\mu_{ik} - X_{ik} - T_{ik})\left(-\frac{\partial T_{ik}}{\partial \beta}\right) =$$

$$\sum_{r=1}^{n}[\overline{B}_{ik}(\psi A_{ir} - \theta B_{ir})\sigma_{rk} - \overline{B}_{ik}(\theta A_{ir} + \psi B_{ir})\beta_{rk}]$$

$$-\sum_{r=1}^{n}\sum_{r'=1}^{n}[\overline{B}_{ik}(A_{ir'}\psi_{rk} - B_{ir'}\theta_{rk})\sigma_{r'r} - \overline{B}_{ik}(A_{ir'}\theta_{rk} + B_{ir'}\psi_{rk})\beta_{r'r}]$$

$$-[\overline{B}_{ik}(\phi L_{ik} - \gamma M_{ik})\sigma - \overline{B}_{ik}(\phi M_{ik} + \gamma L_{ik})\beta] \tag{3.119}$$

$$(\lambda_{ik} - Y_{ik} - U_{ik})\left(-\frac{\partial U_{ik}}{\partial \beta}\right) =$$

$$-\sum_{r=1}^{n}[\overline{A}_{ik}(\theta A_{ir} + \psi B_{ir})\sigma_{rk} + \overline{A}_{ik}(\psi A_{ir} - \theta B_{ir})\beta_{rk}]$$

$$+\sum_{r=1}^{n}\sum_{r'=1}^{n}[\overline{A}_{ik}(A_{ir'}\theta_{rk} + B_{ir'}\psi_{rk})\sigma_{r'r} + \overline{A}_{ik}(A_{ir'}\psi_{rk} - B_{ir'}\theta_{rk})\beta_{r'r}]$$

$$+[\overline{A}_{ik}(\phi M_{ik} + \gamma L_{ik})\sigma + \overline{A}_{ik}(\phi L_{ik} - \gamma M_{ik})\beta] \tag{3.120}$$

Then

$$(\mu_{ik} - X_{ik} - T_{ik})\left(-\frac{\partial T_{ik}}{\partial \beta}\right) + (\lambda_{ik} - Y_{ik} - U_{ik})\left(-\frac{\partial U_{ik}}{\partial \beta}\right)$$

$$= -\sum_{r=1}^{n}[(\theta\overline{A}_{ik} - \psi\overline{B}_{ik})A_{ir} + (\psi\overline{A}_{ik} + \theta\overline{B}_{ik})B_{ir}]\sigma_{rk}$$

$$-\sum_{r=1}^{n}[(\psi\overline{A}_{ik} + \theta\overline{B}_{ik})A_{ir} - (\theta\overline{A}_{ik} - \psi\overline{B}_{ik})B_{ir}]\beta_{rk}$$

$$+\sum_{r=1}^{n}\sum_{r'=1}^{n}[(\overline{A}_{ik}\theta_{rk} - \overline{B}_{ik}\psi_{rk})A_{ir'} + (\overline{A}_{ik}\psi_{rk} + \overline{B}_{ik}\theta_{rk})B_{ir'}]\sigma_{r'r}$$

$$+\sum_{r=1}^{n}\sum_{r'=1}^{n}[(\overline{A}_{ik}\psi_{rk} + \overline{B}_{ik}\theta_{rk})A_{ir'} - (\overline{A}_{ik}\theta_{rk} - \overline{B}_{ik}\psi_{rk})B_{ir'}]\beta_{r'r}$$

$$-[\overline{B}_{ik}\overline{A}_{ik} - \overline{A}_{ik}\overline{B}_{ik}]\sigma + [\overline{A}_{ik}\overline{A}_{ik} + \overline{B}_{ik}\overline{B}_{ik}]\beta \tag{3.121}$$

$$= -\sum_{r=1}^{n}\overline{K}_{ikr}\sigma_{rk} - \sum_{r=1}^{n}\overline{I}_{ikr}\beta_{rk} + \sum_{r=1}^{n}\sum_{r'=1}^{n}\overline{L}_{ikrr'}\sigma_{r'r} + \sum_{r=1}^{n}\sum_{r'=1}^{n}\overline{J}_{ikrr'}\beta_{r'r} \tag{3.122}$$

$$-\overline{N}_{ik}\sigma + \overline{M}_{ik}\beta$$

by using the same notation as in Equations (3.104) and (3.105).

As before, by substituting Equation (3.122) into Equation (3.117) and by using the argument similar to the one used earlier, we obtain

$$\frac{\partial E_m}{\partial \beta} = \sum_{i=1}^{n}\sum_{k=1}^{n} 2\Bigg\{-\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \overline{K}_{rki}\sigma_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \overline{L}_{rr'ki}\sigma_{ik}d\omega\right]$$
$$-\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \overline{I}_{rki}\beta_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \overline{J}_{rr'ki}\beta_{ik}d\omega\right]$$
$$-\left[\int_{\omega_{ik}}^{\omega'_{ik}} \overline{N}_{ik}\sigma d\omega\right] + \left[\int_{\omega_{ik}}^{\omega'_{ik}} \overline{M}_{ik}\beta d\omega\right]\Bigg\} \tag{3.123}$$

Finally, putting it all together as in Equation (3.53), and noting that

$$\frac{\partial E_m}{\partial b_z} = \frac{\partial E_m}{\partial \beta}\frac{\partial \beta}{\partial b_z} = 0 \text{ , } z = 1,3,5,\ldots,q \tag{3.124}$$

Therefore, get,

$$\sum_{i=1}^{n}\sum_{k=1}^{n} 2\Bigg\{-\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \frac{\partial\beta}{\partial b_z}\overline{K}_{rki}\sigma_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \frac{\partial\beta}{\partial b_z}\overline{L}_{rr'ki}\sigma_{ik}d\omega\right]$$
$$-\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \frac{\partial\beta}{\partial b_z}\overline{I}_{rki}\beta_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \frac{\partial\beta}{\partial b_z}\overline{J}_{rr'ki}\beta_{ik}d\omega\right]$$
$$-\left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\beta}{\partial b_z}\overline{N}_{ik}\sigma d\omega\right] + \left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\beta}{\partial b_z}\overline{M}_{ik}\beta d\omega\right]\Bigg\}$$
$$= 0 \text{ , } z = 1,3,5,\ldots,q \tag{3.125}$$

which implies
$$\sum_{i=1}^{n}\sum_{k=1}^{n} \Bigg\{-\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \frac{\partial\beta}{\partial b_z}\overline{K}_{rki}\sigma_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \frac{\partial\beta}{\partial b_z}\overline{L}_{rr'ki}\sigma_{ik}d\omega\right]$$
$$-\sum_{r=1}^{n}\left[\int_{\omega_{rk}}^{\omega'_{rk}} \frac{\partial\beta}{\partial b_z}\overline{I}_{rki}\beta_{ik}d\omega - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \frac{\partial\beta}{\partial b_z}\overline{J}_{rr'ki}\beta_{ik}d\omega\right]$$
$$-\left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\beta}{\partial b_z}\overline{N}_{ik}\hat{\sigma} d\omega\right] + \left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\beta}{\partial b_z}\overline{M}_{ik}\beta d\omega\right]\Bigg\}$$
$$= \sum_{i=1}^{n}\sum_{k=1}^{n}\left[\int_{\omega_{ik}}^{\omega'_{ik}} \frac{\partial\beta}{\partial b_z}\overline{N}_{ik}d\omega\right] = 0 \text{ , } z = 1,3,5,\ldots,q \tag{3.126}$$

where $\hat{\sigma}$ is given as in Equation (3.72).

As an example, take z = 1. Since

$$\frac{\partial\beta}{\partial b_1} = \frac{\partial}{\partial b_1}[b_1\omega - b_3\omega^3 + \cdots] = \omega \tag{3.127}$$

After some algebraic manipulations, Equation (3.126) leads to

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{k=1}^{n}\Bigg\{-\sum_{r=1}^{n}\Bigg[\Bigg(\int_{\omega_{rk}}^{\omega'_{rk}} \omega\overline{K}_{rki} - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \omega\overline{L}_{rr'ki}\Bigg)\,\sigma_{ik}d\omega\Bigg] \\
-\sum_{r=1}^{n}\Bigg[\Bigg(\int_{\omega_{rk}}^{\omega'_{rk}} \omega\overline{I}_{rki} - \sum_{r'=1}^{n}\int_{\omega_{rr'}}^{\omega'_{rr'}} \omega\overline{J}_{rr'ki}\Bigg)\,\beta_{ik}d\omega\Bigg] \\
-\Bigg[\Bigg(\int_{\omega_{ik}}^{\omega'_{ik}} \omega\overline{N}_{ik}\Bigg)\,\hat{\sigma}d\omega\Bigg] + \Bigg[\Bigg(\int_{\omega_{ik}}^{\omega'_{ik}} \omega\overline{M}_{ik}\Bigg)\,\beta d\omega\Bigg]\Bigg\} \\
= \sum_{i=1}^{n}\sum_{k=1}^{n}\Bigg[\int_{\omega_{ik}}^{\omega'_{ik}} \omega\overline{N}_{ik}d\omega\Bigg] = 0
\end{aligned}
\tag{3.128}
$$

By using the same notation as in Equation (3.112) and using σ_{ik}, β_{ik}, σ and β as given by Equation (3.16), we can rewrite Equation (3.128) as follows:

$$
\begin{aligned}
-\sum_{i=1}^{n}\sum_{k=1}^{n}\left[Q_1^{(i,k)}a_0^{(i,k)} - Q_3^{(i,k)}a_2^{(i,k)} + \cdots\right] \\
-\sum_{i=1}^{n}\sum_{k=1}^{n}\left[P_2^{(i,k)}a_1^{(i,k)} - P_4^{(i,k)}a_3^{(i,k)} + \cdots\right] \\
+\left[Z_2b_1 - Z_4b_3 + \cdots\right] = 0
\end{aligned}
\tag{3.129}
$$

We can similarly generate equations similar to Equation (3.129) by taking partial derivatives of E_m with respect to b_z for z = 3,5,7,...,q. These equations can be written in a matrix form, where (3.130)

$$\mathbf{X_4 Y_4 = Z_4}$$

$$
\mathbf{X_4} = \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
\cdots & -Q_1^{(i,k)} & -P_2^{(i,k)} & Q_3^{(i,k)} & P_4^{(i,k)} & \cdots & Z_2 & 0 & -Z_4 & \cdots \\
\cdots & . & . & . & . & \cdots & . & . & . & \cdots \\
\cdots & Q_3^{(i,k)} & P_4^{(i,k)} & -Q_5^{(i,k)} & -P_6^{(i,k)} & \cdots & -Z_4 & 0 & Z_6 & \cdots \\
\ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\tag{3.131}
$$

Table 3.4: The Matrix $\mathbf{X_4}$.

$$
\mathbf{Y}_4 = \begin{bmatrix} \vdots \\ a_0^{(i,k)} \\ a_1^{(i,k)} \\ a_2^{(i,k)} \\ a_3^{(i,k)} \\ \vdots \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}, \quad \mathbf{Z}_4 = \begin{bmatrix} \vdots \\ 0 \\ \cdot \\ 0 \\ \vdots \end{bmatrix} \tag{3.132}
$$

Combining all the matrices in Equations (3.77), (3.95), (3.114) and (3.130), we can regroup them in the following matrix form,

$$
\mathbf{XY} = \mathbf{Z} \text{ or } \mathbf{Y} = \mathbf{X}^{-1}\mathbf{Z} \tag{3.133}
$$

where the matrix $\mathbf{X}$ is given by Table 3.5.

$$
\begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
\cdots & T_0^{(i,k,v,w)} & -U_1^{(i,k,v,w)} & -T_2^{(i,k,v,w)} & \cdots & -R_1^{(v,w)} & S_2^{(v,w)} & R_3^{(v,w)} & \cdots \\
\cdots & U_1^{(i,k,v,w)} & T_2^{(i,k,v,w)} & -U_3^{(i,k,v,w)} & \cdots & -S_1^{(v,w)} & -R_3^{(v,w)} & S_4^{(v,w)} & \cdots \\
\cdots & -T_2^{(i,k,v,w)} & U_3^{(i,k,v,w)} & T_4^{(i,k,v,w)} & \cdots & R_3^{(v,w)} & -S_4^{(v,w)} & -R_5^{(v,w)} & \cdots \\
\cdots & -U_3^{(i,k,v,w)} & -T_4^{(i,k,v,w)} & U_5^{(i,k,v,w)} & \cdots & S_4^{(v,w)} & R_5^{(v,w)} & S_6^{(v,w)} & \cdots \\
\ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
\cdots & -Q_1^{(i,k)} & -P_2^{(i,k)} & Q_3^{(i,k)} & \cdots & Z_2 & 0 & -Z_4 & \cdots \\
\cdots & P_2^{(i,k)} & -Q_3^{(i,k)} & -P_4^{(i,k)} & \cdots & 0 & Z_4 & 0 & \cdots \\
\cdots & Q_3^{(i,k)} & P_4^{(i,k)} & -Q_5^{(i,k)} & \cdots & -Z_4 & 0 & Z_6 & \cdots \\
\ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \tag{3.134}
$$

Table 3.5: The matrix X

and

$$\mathbf{Y} = \begin{bmatrix} \vdots \\ a_0^{(i,k)} \\ a_1^{(i,k)} \\ a_2^{(i,k)} \\ \vdots \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} \vdots \\ S_0^{(v,w)} \\ -R_1^{(v,w)} \\ -S_2^{(v,w)} \\ \vdots \\ 0 \\ Z_2 \\ 0 \\ \vdots \end{bmatrix} \tag{3.135}$$

Thus the matrix Equation (3.133) determines the unknown controller parameters $a_r^{(i,k)}$ and b_z for $0 \le r \le p(i,k)$, $1 \le z \le q$ and $1 \le i,k \le n$.

As an illustration, for a system of order $n = 2$, a second order controller with unknown parameters a's and b's, is given as

$$\mathrm{C}(s) = \frac{\begin{bmatrix} a_0^{(1,1)} + a_1^{(1,1)}s + a_2^{(1,1)}s^2 & a_0^{(1,2)} + a_1^{(1,2)}s + a_2^{(1,2)}s^2 \\ a_0^{(2,1)} + a_1^{(2,1)}s + a_2^{(2,1)}s^2 & a_0^{(2,2)} + a_1^{(2,2)}s + a_2^{(2,2)}s^2 \end{bmatrix}}{1 + b_1 s + b_2 s^2} \tag{3.136}$$

For this case, the matrices Y and Z are given by Equation (3.137) and the matrix X is given by (3.138) (Table 3.6).

$$\mathbf{Y} = \begin{bmatrix} a_0^{(1,1)} \\ a_1^{(1,1)} \\ a_2^{(1,1)} \\ a_0^{(1,2)} \\ a_1^{(1,2)} \\ a_2^{(1,2)} \\ a_0^{(2,1)} \\ a_1^{(2,1)} \\ a_2^{(2,1)} \\ a_0^{(2,2)} \\ a_1^{(2,2)} \\ a_2^{(2,2)} \\ b_1 \\ b_2 \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} S_0^{(1,1)} \\ -R_1^{(1,1)} \\ -S_2^{(1,1)} \\ S_0^{(1,2)} \\ -R_1^{(1,2)} \\ -S_2^{(1,2)} \\ S_0^{(2,1)} \\ -R_1^{(2,1)} \\ -S_2^{(2,1)} \\ S_0^{(2,2)} \\ -R_1^{(2,2)} \\ -S_2^{(2,2)} \\ 0 \\ Z_2 \end{bmatrix} \tag{3.137}$$

$$
\begin{bmatrix}
T_0^{(1,1,1,1)} & -U_1^{(1,1,1,1)} & -T_2^{(1,1,1,1)} & T_0^{(1,2,1,1)} & -U_1^{(1,2,1,1)} & -T_2^{(1,2,1,1)} & T_0^{(2,1,1,1)} & -U_1^{(2,1,1,1)} & -T_2^{(2,1,1,1)} & T_0^{(2,2,1,1)} & -U_1^{(2,2,1,1)} & -T_2^{(2,2,1,1)} & -R_1^{(1,1)} & S_2^{(1,1)} \\
U_1^{(1,1,1,1)} & T_2^{(1,1,1,1)} & -U_3^{(1,1,1,1)} & U_1^{(1,2,1,1)} & T_2^{(1,2,1,1)} & -U_3^{(1,2,1,1)} & U_1^{(2,1,1,1)} & T_2^{(2,1,1,1)} & -U_3^{(2,1,1,1)} & U_1^{(2,2,1,1)} & T_2^{(2,2,1,1)} & -U_3^{(2,2,1,1)} & -S_2^{(1,1)} & -R_3^{(1,1)} \\
-T_2^{(1,1,1,1)} & U_3^{(1,1,1,1)} & T_4^{(1,1,1,1)} & -T_2^{(1,2,1,1)} & U_3^{(1,2,1,1)} & T_4^{(1,2,1,1)} & -T_2^{(2,1,1,1)} & U_3^{(2,1,1,1)} & T_4^{(2,1,1,1)} & -T_2^{(2,2,1,1)} & U_3^{(2,2,1,1)} & T_4^{(2,2,1,1)} & R_3^{(1,1)} & -S_4^{(1,1)} \\
T_0^{(1,1,1,2)} & -U_1^{(1,1,1,2)} & -T_2^{(1,1,1,2)} & T_0^{(1,2,1,2)} & -U_1^{(1,2,1,2)} & -T_2^{(1,2,1,2)} & T_0^{(2,1,1,2)} & -U_1^{(2,1,1,2)} & -T_2^{(2,1,1,2)} & T_0^{(2,2,1,2)} & -U_1^{(2,2,1,2)} & -T_2^{(2,2,1,2)} & -R_1^{(1,2)} & S_2^{(1,2)} \\
U_1^{(1,1,1,2)} & T_2^{(1,1,1,2)} & -U_3^{(1,1,1,2)} & U_1^{(1,2,1,2)} & T_2^{(1,2,1,2)} & -U_3^{(1,2,1,2)} & U_1^{(2,1,1,2)} & T_2^{(2,1,1,2)} & -U_3^{(2,1,1,2)} & U_1^{(2,2,1,2)} & T_2^{(2,2,1,2)} & -U_3^{(2,2,1,2)} & -S_2^{(1,2)} & -R_3^{(1,2)} \\
-T_2^{(1,1,1,2)} & U_3^{(1,1,1,2)} & T_4^{(1,1,1,2)} & -T_2^{(1,2,1,2)} & U_3^{(1,2,1,2)} & T_4^{(1,2,1,2)} & -T_2^{(2,1,1,2)} & U_3^{(2,1,1,2)} & T_4^{(2,1,1,2)} & -T_2^{(2,2,1,2)} & U_3^{(2,2,1,2)} & T_4^{(2,2,1,2)} & R_3^{(1,2)} & -S_4^{(1,2)} \\
T_0^{(1,1,2,1)} & -U_1^{(1,1,2,1)} & -T_2^{(1,1,2,1)} & T_0^{(1,2,2,1)} & -U_1^{(1,2,2,1)} & -T_2^{(1,2,2,1)} & T_0^{(2,1,2,1)} & -U_1^{(2,1,2,1)} & -T_2^{(2,1,2,1)} & T_0^{(2,2,2,1)} & -U_1^{(2,2,2,1)} & -T_2^{(2,2,2,1)} & -R_1^{(2,1)} & S_2^{(2,1)} \\
U_1^{(1,1,2,1)} & T_2^{(1,1,2,1)} & -U_3^{(1,1,2,1)} & U_1^{(1,2,2,1)} & T_2^{(1,2,2,1)} & -U_3^{(1,2,2,1)} & U_1^{(2,1,2,1)} & T_2^{(2,1,2,1)} & -U_3^{(2,1,2,1)} & U_1^{(2,2,2,1)} & T_2^{(2,2,2,1)} & -U_3^{(2,2,2,1)} & -S_2^{(2,1)} & -R_3^{(2,1)} \\
-T_2^{(1,1,2,1)} & U_3^{(1,1,2,1)} & T_4^{(1,1,2,1)} & -T_2^{(1,2,2,1)} & U_3^{(1,2,2,1)} & T_4^{(1,2,2,1)} & -T_2^{(2,1,2,1)} & U_3^{(2,1,2,1)} & T_4^{(2,1,2,1)} & -T_2^{(2,2,2,1)} & U_3^{(2,2,2,1)} & T_4^{(2,2,2,1)} & R_3^{(2,1)} & -S_4^{(2,1)} \\
T_0^{(1,1,2,2)} & -U_1^{(1,1,2,2)} & -T_2^{(1,1,2,2)} & T_0^{(1,2,2,2)} & -U_1^{(1,2,2,2)} & -T_2^{(1,2,2,2)} & T_0^{(2,1,2,2)} & -U_1^{(2,1,2,2)} & -T_2^{(2,1,2,2)} & T_0^{(2,2,2,2)} & -U_1^{(2,2,2,2)} & -T_2^{(2,2,2,2)} & -R_1^{(2,2)} & S_2^{(2,2)} \\
U_1^{(1,1,2,2)} & T_2^{(1,1,2,2)} & -U_3^{(1,1,2,2)} & U_1^{(1,2,2,2)} & T_2^{(1,2,2,2)} & -U_3^{(1,2,2,2)} & U_1^{(2,1,2,2)} & T_2^{(2,1,2,2)} & -U_3^{(2,1,2,2)} & U_1^{(2,2,2,2)} & T_2^{(2,2,2,2)} & -U_3^{(2,2,2,2)} & -S_2^{(2,2)} & -R_3^{(2,2)} \\
-T_2^{(1,1,2,2)} & U_3^{(1,1,2,2)} & T_4^{(1,1,2,2)} & -T_2^{(1,2,2,2)} & U_3^{(1,2,2,2)} & T_4^{(1,2,2,2)} & -T_2^{(2,1,2,2)} & U_3^{(2,1,2,2)} & T_4^{(2,1,2,2)} & -T_2^{(2,2,2,2)} & U_3^{(2,2,2,2)} & T_4^{(2,2,2,2)} & R_3^{(2,2)} & -S_4^{(2,2)} \\
-Q_1^{(1,1)} & -P_2^{(1,1)} & Q_3^{(1,1)} & -Q_1^{(1,2)} & -P_2^{(1,2)} & Q_3^{(1,2)} & -Q_1^{(2,1)} & -P_2^{(2,1)} & Q_3^{(2,1)} & -Q_1^{(2,2)} & -P_2^{(2,2)} & Q_3^{(2,2)} & Z_2 & 0 \\
P_2^{(1,1)} & -Q_3^{(1,1)} & -P_4^{(1,1)} & P_2^{(1,2)} & -Q_3^{(1,2)} & -P_4^{(1,2)} & P_2^{(2,1)} & -Q_3^{(2,1)} & -P_4^{(2,1)} & P_2^{(2,2)} & -Q_3^{(2,2)} & -P_4^{(2,2)} & 0 & Z_4 \\
Q_3^{(1,1)} & P_4^{(1,1)} & -Q_5^{(1,1)} & Q_3^{(1,2)} & P_4^{(1,2)} & -Q_5^{(1,2)} & Q_3^{(2,1)} & P_4^{(2,1)} & -Q_5^{(2,1)} & Q_3^{(2,2)} & P_4^{(2,2)} & -Q_5^{(2,2)} & -Z_4 & 0
\end{bmatrix}
\tag{3.138}
$$

Table 3.6: The matrix X

3.4 Remarks

Before we conclude this chapter, we wish to make some explanatory comments on the technique.

1. For using the technique proposed in this chapter, it is assumed that the given plant is, at least, marginally stable or if unstable, it has already been, somehow, made at least marginally stable.
2. The 'desired' closed loop transfer function matrix has to be mathematically formulated from the design specifications.
3. Observe that the integrand in Equation (3.11) for each fixed i and k, is the difference between two polynomials in ω with coefficients involving a's and b's. Note that when we equate to zero the partial derivatives of E_m with respect to a's and b's, we will get linear algebraic equations in a's and b's.
4. Since the integrand in E_m, for each fixed i and k, is the difference between two polynomials in ω, intuitively, it seems logical to conclude that the best 'fit' will be achieved when the polynomials in ω are of the same degree unless the higher degree terms in one of them compared to the other have relatively small coefficients. We formalize this intuitive idea in the proposed algorithm.
5. In order to check whether the integrand in E_m for each fixed i and k has the best 'fit' as described previously, we can just check the pole–zero deficiency between each entry of the compensated transfer function matrix and the 'desired' transfer function matrix.
6. For a system of order, n = 2, suppose the degree of the plant, controller and the 'desired' transfer function matrices are given as follows:

$$\delta\{g_{ik}(s)\} = m_{ik}, \quad 1 \leq i,\ k \leq 2$$

$$\delta\{g(s)\} = n$$

where $m_{ik} \le n$, for $1 \le i, k \le 2$.

$$\delta\{c_{ik}(s)\} = p_{ik}, \quad 1 \le i, \; k \le 2$$

$$\delta\{a(s)\} = q$$

where $p_{ik} \le q$, for $1 \le i, k \le 2$.

$$\delta\{d_{ik}(s)\} = \ell_{ik}, \quad 1 \le i, \; k \le 2$$

$$\delta\{d(s)\} = k$$

where $\ell_{ik} \le k$, for $1 \le i, k \le 2$.

In order for the integrand in E_m to have the best 'fit', one would like the following conditions to be satisfied as far as possible:

$$\max\{m_{11} + p_{11}, \; m_{12} + p_{21}\} + k$$
$$= \max\{n + q + \ell_{11}, m_{11} + p_{12} + \ell_{21}, \; m_{12} + p_{22} + \ell_{21}\}$$
$$\max\{m_{11} + p_{12}, \; m_{12} + p_{22}\} + k$$
$$= \max\{n + q + \ell_{12}, \; m_{11} + p_{12} + \ell_{22}, \; m_{12} + p_{22} + \ell_{22}\}$$
$$\max\{m_{21} + p_{11}, \; m_{22} + p_{21}\} + k$$
$$= \max\{n + q + \ell_{21}, \; m_{21} + p_{11} + \ell_{11}, \; m_{22} + p_{21} + \ell_{11}\}$$
$$\max\{m_{21} + p_{12}, \; m_{22} + p_{22}\} + k$$
$$= \max\{n + q + \ell_{22}, \; m_{21} + p_{11} + \ell_{12}, \; m_{22} + p_{21} + \ell_{12}\}$$

7. We can enhance the 'desired' transfer function matrix D(s) so that both the polynomials in the integrand of E_m, for each fixed i and k, are of the same degree by adding as many relatively nondominant poles in the entries of D(s) as needed in such a way that the steady state response stays unaffected. This approach of enhancing the desired transfer function will lead to better minimizing of E_m without adding to the order of the controller.

8. If the coefficients in the plant and/or 'desired' transfer function matrix are large, then the minimization of E_m will lead to overflow and/or numerical errors in the integration, matrix inversion etc. Thus, to avoid this, it's advisable to divide all the coefficients of the transfer function matrix by a suitable factor to make them small for computation. Evidently, this has no impact on the algorithm.

9. The frequency interval of interest $[\omega_{ik},\omega_{ik}']$ is usually the bandwidth of the $(i,k)^{ik}$ element in the 'desired' transfer function matrix D(s) and, in general, one gets good results by minimizing E_m over this frequency interval. But, in some cases, if the bandwidth is large and/or the coefficients in the transfer function matrix involved are also large, then the minimization of E_m over the entire bandwidth leads to large entries in matrix **X** and vector **Z**, (Equations (3.138) and (3.137)), respectively. This leads to overflow and numerical errors in the integration, matrix inversion, etc. In order to avoid this, it is advisable to minimize E_m over smaller intervals than the bandwidths, usually one half to one quarter time the bandwidths. One may have to tinker a bit with the limits of integration for E_m within the above guidelines until satisfactory results are obtained. This should not be very time consuming as the algorithm is computer–aided.

10. A computer code in Fortran 77 has been developed to implement the algorithm with the help of IMSL routines. The program has been written to be user–friendly. The listing of the program is given in Appendix A.

11. The technique is very easy to use overall. All that the user has to do is to input the parameters of the plant, feedback element and 'desired' transfer function matrix, the order of the controller (one usually starts with the zero order), and the limits of the integration for E_m. As output, he gets the parameters of the controller and also, if the system has two inputs and two

outputs, he also gets the compensated closed loop transfer function with its poles and zeros of the elements of the transfer function matrix, and the frequency responses of the 'desired' and the compensated systems. If the 'fit' is good, one is done. Otherwise, the user increases the order of the controller and/or alters the limits of integration for E_m as indicated above and continues until satisfactory results are obtained.

12. Finally, it should be emphasized that this simple algorithm would not solve all the problems of multivariable control systems design satisfactorily. This will only give a candidate for a preliminary design which will have to be fine-tuned to make it workable. But it gives an excellent starting point for design purposes.

3.5 Summary

In this chapter, a computer–aided technique of designing a multivariable control system is proposed. The controller parameters are obtained by matching the frequency responses of the compensated closed loop system and a 'desired' closed loop system over a certain frequency interval of interest. The technique is easy to use as the controller parameters turn out to be the solutions of linear algebraic equations. Finally, some explanatory remarks were given of the algorithm and the overall technique.

4. ILLUSTRATIVE EXAMPLES

Two numerical examples are presented. First, we reexamine the example of Chen [1] and compare the controllers designed by the proposed technique and Chen's technique. The second one pertains to the design of the control system for the YF–16 CCV in its longitudinal axis.

4.1 Example 1 [1]

For Chen's example [1], the block diagram for a two–input two–output control system is given by Figure 3.1 in Chapter 3 with the feedback element, $\mathbf{H}(s) = \mathbf{I}_2$, and the transfer function matrix of the plant given as

$$\mathbf{G}(s) = \frac{\begin{bmatrix} 1.265s + 2.122 & 941.479s - 1506.997 \\ 7.190s + 10.238 & 1241.0169s - 2107.295 \end{bmatrix}}{s^2 + 2.918s + 2.107} \tag{4.1}$$

It is desired to synthesize a controller, $\mathbf{C}(s)$ so that the overall closed loop system matches the 'desired' closed loop system. The 'desired' closed loop transfer function as synthesized by Chen [1] to meet the specification is given by

$$\mathbf{D}(s) = \frac{\begin{bmatrix} 660.2308s + 379289.8456 & 11.9093s \\ -371.6217s & 565.843s + 378761 \end{bmatrix}}{s^2 + 1229.2923s + 380040.6048} \tag{4.2}$$

From the magnitude of the frequency response plots of the elements of D(s), the frequency interval $[\omega_{ik}, \omega'_{ik}] = [50,100]$ for $1 \leq i, k \leq 2$ is selected over which the error function E_m will be minimized. Using Equation (3.129), the zero order controller is obtained as

$$\mathbf{C}_0(s) = \begin{bmatrix} -212.5080 & 93.2877 \\ 1.0392 & -0.1393 \end{bmatrix} \tag{4.3}$$

Therefore, the compensated closed loop transfer function matrix is obtained as

$$\mathbf{F}_0(s) = [\mathbf{I}_2 + \mathbf{G}(s)\mathbf{C}_0(s)\mathbf{H}(s)]^{-1}\mathbf{G}(s)\mathbf{C}_0(s) \tag{4.4}$$

$$= \frac{\begin{bmatrix} f^0_{11}(s) & f^0_{12}(s) \\ f^0_{21}(s) & f^0_{22}(s) \end{bmatrix}}{f^0(s)} \tag{4.5}$$

where

$$\begin{aligned}
f^0_{11}(s) &= 709.5624(s + 1.3102)(s + 1.6084)(s + 495.0238) \\
f^0_{12}(s) &= -13.1391(s + 0.9109)(s + 1.3118)(s + 1.6062) \\
f^0_{21}(s) &= -238.268(s - 0.0598)(s + 1.3118)(s + 1.6062) \\
f^0_{22}(s) &= 497.865(s + 1.3102)(s + 1.6084)(s + 704.6024) \\
f^0(s) &= (s + 1.3102)(s + 1.6084)(s + 485.5063)(s + 724.8383)
\end{aligned}$$

The frequency responses of each element of the compensated and the 'desired' closed loop transfer function matrices are given in Figures 4.1 through 4.8. In addition, the time responses to the inputs $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ of the compensated and the 'desired' systems are given in Figures 4.9 and 4.10.

The zero order controller designed by Chen [1] is given by

$$\mathbf{K}(s) = \begin{bmatrix} -202.0704 & 96.3172 \\ 0.9815 & -0.1356 \end{bmatrix} \tag{4.6}$$

The compensated closed loop transfer function matrix is given by

$$\mathbf{F}_k(s) = [\mathbf{I}_2 + \mathbf{G}(s)\mathbf{K}(s)\mathbf{H}(s)]^{-1}\mathbf{G}(s)\mathbf{K}(s) \tag{4.7}$$

$$= \frac{\begin{bmatrix} f_{11}^k(s) & f_{12}^k(s) \\ f_{21}^k(s) & f_{22}^k(s) \end{bmatrix}}{f^k(s)} \tag{4.8}$$

where

$$f_{11}^k(s) = 668.4426(s + 1.3102)(s + 1.6084)(s + 523.7637)$$

$$f_{12}^k(s) = -5.8233(s - 0.006235)(s + 1.3118)(s + 1.6062)$$

$$f_{21}^k(s) = -234.828(s + 0.002073)(s + 1.3118)(s + 1.6062)$$

$$f_{22}^k(s) = 524.2388(s + 1.3102)(s + 1.6084)(s + 667.1694)$$

$$f^k(s) = (s + 1.3102)(s + 1.6084)(s + 516.8301)(s + 678.7686)$$

For the purpose of comparison, the frequency response of each element of Chen's compensated closed loop transfer function matrix are superimposed on Figures 4.1 through 4.8. In addition, the time responses to the inputs $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ of Chen's compensated system are also *given* in Figures 4.9 and 4.10. It can be observed from Figure 4.3, that the magnitude response of the (2,1) element and from Figure 4.6, that the phase response of the (1,2) element, of the compensated closed loop transfer function matrix have poor match with the magnitude and the phase responses for the corresponding elements in the desired transfer function matrix. Therefore, it is desirable to improve this particular mismatch.

Now to improve the match with the desired transfer function matrix, we consider a first order controller matrix. Again, using Equation (3.129) with the same frequency interval of interest, the first order controller is obtained as

$$\mathbf{C}_1(s) = \frac{\begin{bmatrix} -224.879(s + 1.641) & 99.618(s + 1.575) \\ 1.00343(s + 1.824) & -0.1212(s + 1.831) \end{bmatrix}}{(s + 1.836)} \tag{4.9}$$

Therefore, the compensated closed loop transfer function matrix is

$$\mathbf{F}_1(s) = [\mathbf{I}_2 + \mathbf{G}(s)\mathbf{C}_1(s)\mathbf{H}(s)]^{-1}\mathbf{G}(s)\mathbf{C}_1(s) \tag{4.10}$$

$$= \frac{\begin{bmatrix} f_{11}^1(s) & f_{12}^1(s) \\ f_{21}^1(s) & f_{22}^1(s) \end{bmatrix}}{f^1(s)} \tag{4.11}$$

where

$$f_{11}^1(s) = 660.236(s + 1.3102)(s + 1.5494)(s + 1.6084)(s + 1.8223)(s + 574.4794)$$

$$f_{12}^1(s) = -11.909(s - 0.07757)(s + 1.3118)(s + 1.6062)(s + 1.6131)(s + 1.836)$$

$$f_{21}^1(s) = -371.606(s - 0.1277)(s + 1.3118)(s + 1.6062)(s + 1.6608)(s + 1.836)$$

$$f_{22}^1(s) = 565.8422(s + 1.3102)(s + 1.5494)(s + 1.6084)(s + 1.8222)(s + 669.379)$$

$$f^1(s) = (s + 1.3102)(s + 1.5494)(s + 1.6084)(s + 1.8222)$$

$$\times(s^2 + 1229.2962s + 380042.6328)$$

Figures 4.11 through 4.20 show the comparison of the frequency responses and the unit step time responses of the compensated system and the 'desired' system. These figures show that all the magnitude, phase and time responses of the compensated closed loop system match very well with the 'desired' closed loop system. Comparison of Figures 4.1 through 4.10 and Figures 11 through 20 shows that the results obtained by the first order compensation by the proposed technique are much better.

4.2 Example 2

A preliminary longitudinal control systems design of YF–16 CCV airframe is presented using the frequency matching technique. The technique is used to choose the parameters of the controller to match closely a 'desired' closed loop frequency response. The 'desired' closed loop frequency response is based on meeting the military flying qualities specification, MIL–F–8785C [12,21], for Level 1 flying qualities.

4.2.1 Airframe Model

The airframe model chosen is the YF–16 CCV aircraft in conventional flight at 0.8 Mach at 20,000 feet [7]. The following state space model was provided by Mr. Tom Gentry, the Project Engineer who derived it from [7]. Based on these aircraft longitudinal data, the state space model is given as (Figure 4.21):

$$\dot{\underline{x}} = \mathbf{A_1}\underline{x} + \mathbf{B_1}\underline{u} \tag{4.12}$$

$$\underline{y} = \mathbf{C_1}\underline{x} + \mathbf{D_1}\underline{u} \tag{4.13}$$

where

$$\mathbf{A_1} = \begin{bmatrix} -0.016484 & 111.35 & -32.17 & 0 \\ -9.3359E-5 & -1.0046 & 0 & 0.9983 \\ 0 & 0 & 0 & 1 \\ 2.6976E-5 & 11.814 & 0 & -1.1174 \end{bmatrix} \tag{4.14}$$

$$\mathbf{B_1} = \begin{bmatrix} 6.648 & 8.662 \\ 0.15306 & 0.16445 \\ 0 & 0 \\ 21.096 & 4.9075 \end{bmatrix} \tag{4.15}$$

$$\mathbf{C_1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.16}$$

and

$$\mathbf{D_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{4.17}$$

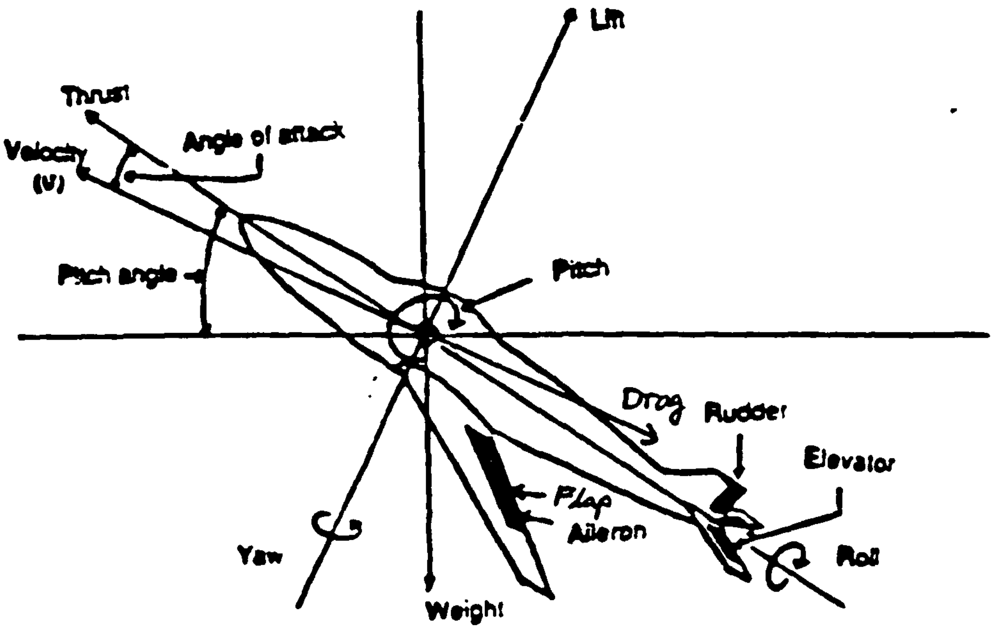

Figure 4.21: Forces and moments on an airplane in a steady climb

$$
\underline{x} = \begin{bmatrix} u \\ \alpha \\ \theta \\ q \end{bmatrix}
\begin{array}{l} \text{forward speed, } ft/sec \\ \text{angle of attack, } rad \\ \text{pitch attitude, } rad \\ \text{pitch rate, } rad/sec \end{array}
$$

$$
\underline{u} = \begin{bmatrix} \delta_e \\ \delta_f \end{bmatrix}
\begin{array}{l} \text{horizontal tail deflection, } rad \\ \text{flap deflection, } rad \end{array}
$$

$$
\underline{y} = \begin{bmatrix} \alpha \\ q \end{bmatrix}
\begin{array}{l} \text{angle of attack, } rad \\ \text{pitch rate, } rad/sec \end{array}
$$

We should point out that the airframe is unstable with an unstable pole at s = +2.372.

4.2.2 Actuator Model

For preliminary design a simple actuator model is desired, but it should still be reasonably accurate. For this example, the actuator model represented in state space form is as follows:

$$
\dot{\underline{z}} = \mathbf{A_2}\underline{z} + \mathbf{B_2}\underline{v} \tag{4.18}
$$

$$
\underline{w} = \mathbf{C_2}\underline{z} + \mathbf{D_2}\underline{v} \tag{4.19}
$$

where

$$
\mathbf{A_2} = \begin{bmatrix} -13 & 0 \\ 0 & -13 \end{bmatrix} \tag{4.20}
$$

$$\mathbf{B_2} = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} \tag{4.21}$$

$$\mathbf{C_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.22}$$

$$\mathbf{D_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{4.23}$$

and

$$\underline{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \begin{matrix} \text{actuator state} \\ \text{actuator state} \end{matrix}$$

$$\underline{v} = \begin{bmatrix} \delta'_e \\ \delta'_f \end{bmatrix} \begin{matrix} \text{commanded horizontal tail deflection, } rad \\ \text{commanded flap deflection, } rad \end{matrix}$$

$$\underline{w} = \begin{bmatrix} \delta_e \\ \delta_f \end{bmatrix} \begin{matrix} \text{horizontal tail deflection, } rad \\ \text{flap deflection, } rad \end{matrix}$$

For this example this actuator model is used to relate the actual surface deflection vector, $[\delta_e, \delta_f]^T$, to the commanded surface deflection vector, $[\delta'_e, \delta'_f]^T$.

4.2.3 Airframe–Plus–Actuator Model

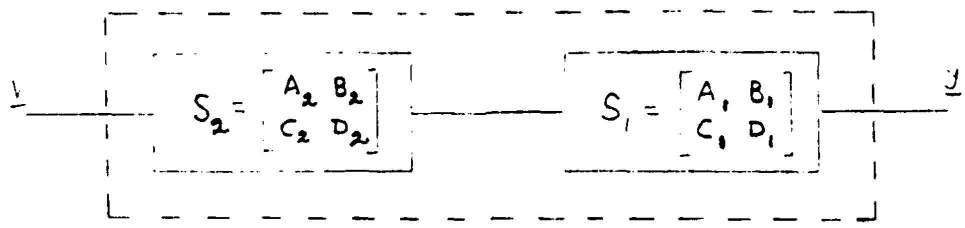

Figure 4.22: The airframe–plus–actuator model

The actuator model is inserted in series with the airframe model, as shown in Figure 4.22, calling it the airframe–plus–actuator model, which is the plant where v is the two–input vector and y is the two–output vector. Denote the plant, the airframe-plus–actuator model, by the state space representation as $S = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$, where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_2 & \mathbf{0} \\ \mathbf{B}_1\mathbf{C}_2 & \mathbf{A}_1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{B}_1\mathbf{D}_2 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{D}_1\mathbf{C}_2 & \mathbf{C}_1 \end{bmatrix} \quad \mathbf{D} = \mathbf{D}_1\mathbf{D}_2 \tag{4.24}$$

Thus the plant in transfer function matrix form is given by

$$\mathbf{P}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D} \tag{4.25}$$

$$= \frac{\begin{bmatrix} p_{11}(s) & p_{12}(s) \\ p_{21}(s) & p_{22}(s) \end{bmatrix}}{p(s)} \tag{4.26}$$

where

$$p_{11}(s) = 1.9898s^3 + 276.03s^2 + 4.543s + 0.82539$$

$$p_{12}(s) = 2.1378s^3 + 66.103s^2 + 1.0805s + 0.19346$$

$$p_{21}(s) = 274.25s^3 + 303.54s^2 + 7.6929s$$

$$p_{22}(s) = 63.797s^3 + 90.402s^2 + 2.0213s$$

$$p(s) = s^5 + 15.138s^4 + 17.174s^3 - 138.3s^2 - 2.1981s - 0.44993$$

Here the input is $[\delta_e', \delta_f']$ and the output is $[\alpha, q]^T$. We would like to point out that the plant checks out to be both controllable and observable.

4.2.4 Stabilization of the plant

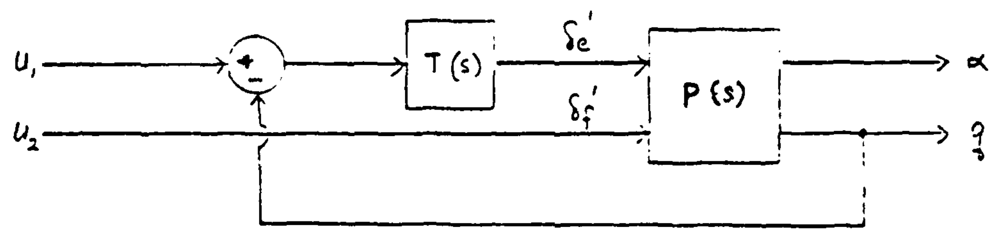

Figure 4.23: Block diagram of the stabilised plant

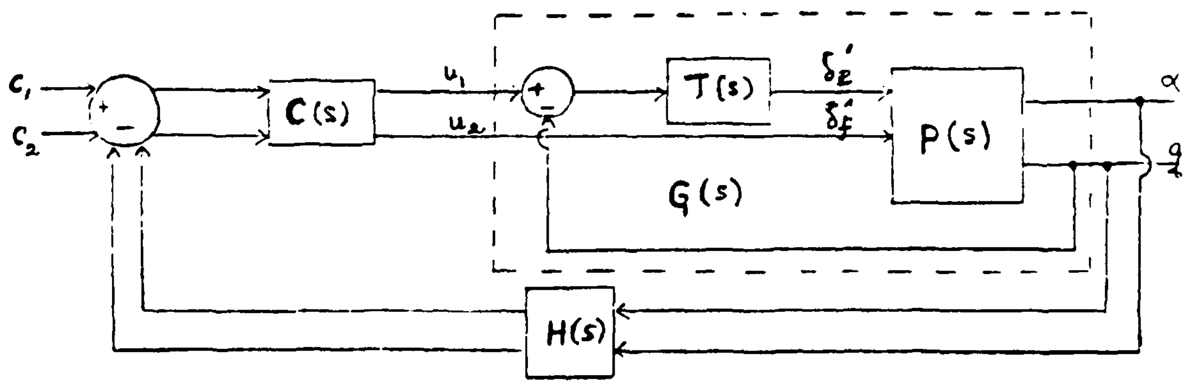

Figure 4.24: The control system design of YF–16 CCV

Since the proposed technique in this report requires the plant to be stable, it is necessary to stabilize the airframe–plus–actuator model with an inner loop. Here we choose to feed back sensed pitch rate, q only, because the measurement of the pitch rate is much more accurate than of the angle of attack. This introduces an additional control element in the design. The inner loop control element was placed in the forward path in series with the airframe–plus–actuator as shown in Figure 4.23. By a careful analysis of the pole–zero pattern of the plant, a stabilizing controller T(s) was designed, which is given by

$$T(s) = \frac{5s^2 + 0.0813s + 0.11283}{s^2 + 0.02s} \tag{4.27}$$

This controller is shown in the block diagram in Figure 4.23. After doing some routine block diagram manipulation, the resultant stabilized plant comes out to be

$$G(s) = \frac{\begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix}}{g(s)} \tag{4.28}$$

where

$$\begin{aligned}
g_{11}(s) &= 9.9489s^6 + 1509.6s^5 + 17989s^4 + 625.59s^3 + 463.91s^2 + 7.6291s + 1.2107 \\
g_{12}(s) &= 2.1379s^6 + 93.937s^5 + 3159.1s^4 + 94.808s^3 + 55.052s^2 + 0.63732s \\
g_{21}(s) &= 1371.2s^6 + 19366s^5 + 20114s^4 + 1258s^3 + 454.23s^2 + 11.284s \\
g_{22}(s) &= 63.797s^6 + 921.05s^5 + 1195.6s^4 + 49.822s^3 + 0.52553s^2 \\
g(s) &= s^8 + 28.158s^7 + 1585.8s^6 + 19455s^5 + 18316s^4 + 1193s^3 + 447.8s^2 + 11.167s
\end{aligned}$$

Here the input is $[u_1,u_2]^T$ and the output is $[\alpha,q]^T$. Now, with the stabilized plant, G(s) and the feedback element taken as $\mathbf{H}(s) = I_2$, we can use the proposed design technique to design a controller C(s) as shown in Figure 4.24.

4.2.5 Design Requirements

Before we use the proposed design technique, it is necessary to specify the 'desired' closed loop transfer function matrix D(s). The desired dynamics for the angle of attack, α, and pitch rate, q, to pilot command, c, for the flight control system are given as follows:

$$\frac{\alpha}{c} = \frac{21.2}{s^2 + 7s + 25} \tag{4.29}$$

$$\frac{q}{c} = \frac{21.1(s + 1.08)}{s^2 + 7s + 25} \tag{4.30}$$

All the specifications for different cases for this example were provided to us by Mr. Tom Gentry, the Project Engineer. We wish to give some explanation of the above desired dynamics based on the flying qualities of this aircraft which are well documented in [4] and our description below is taken from the above paper.

MIL–F–8785C [21] is used to determine the basic dynamic requirements. We concentrate here on the short–period and time delay requirements to achieve desired performance (Level 1). For a given aircraft Class (here Class IV, fighter) and Flight Phase Category (here Category A, precise control and gross maneuvering up–and–away) the appropriate equivalent short–period requirements (Sections 3.2.2.1.1 and 3.2.2.1.2) and equivalent time delay requirement (Section 3.5.3) for Level 1 flying qualities are determined. These equivalent–system parameters are to be obtained by matching classical responses to the actual pitch–rate and normal–acceleration transfer functions. The section on equivalent short–period natural frequency also requires the use of the parameter n/α in g's per rad. This parameter is the ratio of the change in a steady–state normal load factor $(-a_z/g)$ per unit change in angle of attack. These are dynamic requirements which we will use to define the dynamic responses of the design.

At 0.8 Mach and 20,000 feet altitude, air–to–air combat is an appropriate task. This task is in Flight Phase Category A in MIL–F–8785C. The factor n/α may be

estimated from the larger of the two non–zero numerator rate to elevator deflection transfer function. Here this factor is 1.082. $1/T_{\theta_2}$, is approximately equal to L_α, the dimensional derivative. From [8], for example,

$$\frac{n}{\alpha} \approx \frac{V}{g} \cdot \frac{1}{T_{\theta_2}} \tag{4.31}$$

where V is the aircraft speed in feet per second and g is the acceleration due to gravity in ft/sec^2. Here V is 829.6 ft/sec and g is 32.17 ft/sec^2. The parameter n/α is estimated to be 27.9 g's/rad.

Based on this value of n/α, Figure 1 in MIL–F–8785C for Flight Phase Category A shows an allowable range for Level 1 flying qualities of 0.28 and 3.6 for the parameter $\omega_{sp}^2/(n/\alpha)$. This parameter is referred to as the Control Anticipation Parameter (CAP). For $n/\alpha = 27.9$ and the allowable range of CAP, an allowable range of equivalent short-period natural frequency, ω_{sp}, is 2.79 to 10 rad/sec. A design value of 5 rad/sec was chosen here as being a conservative choice to allow for changes up or down and still be in compliance with the specification.

The allowable range of equivalent short–period damping ratio is found in Table IV in MIL–F–8785C. For Flight Phase Category A, Level 1 flying qualities, this table permits a damping ratio of 0.35 to 1.30. A value of 0.7 is chosen as a reasonable value for the design. This value is also typical of many real aircraft.

The last parameter to be specified is equivalent time delay. For Level 1 flying qualities in all Flight Phase Categories, the maximum equivalent time delay allowed is 0.1 sec. in Table XIV [21]. Equivalent time delay is the total of pure time delay plus the contributions of high–frequency elements such as actuators, compensation filters, etc. in the equivalent system match of the aircraft's actual dynamics. The intent of this requirement is to minimize equivalent time delay, so for our model response we will take a value of 0 (zero).

Thus, based on the task requirement and the aircraft class, the design parameters are an equivalent short–period natural frequency of 5 rad/sec, a damping ratio of 0.7 and no equivalent time delay.

There is now one last detail of the design requirements to specify, which is to completely define the exact dynamic specification for the aircraft response of interest for the design process to follow. This is not always as straightforward as it may seem. The two model parameters previously specified, short–period natural frequency and damping ratio, only define the denominator dynamics of the short–period approximation to the aircraft dynamics. The numerator dynamics must be specified also. For short–period dynamics, this requires that the pitch rate numerator, for example, be first–order, which for a conventional aircraft is the factor at $1/T_{\theta_2}$. As we have noted, $1/T_{\theta_2}$ is largely determined by L_α, which in turn is determined by the choice of wing size and planform. Here we assume that this choice has been made to satisfy other design requirements and will check its flying qualities implications.

All of the longitudinal responses are coupled through the airframe. Because only one control surface is available to control the longitudinal responses of the aircraft, complete specification of one response, such as pitch rate or normal acceleration, completely specifies all other responses as coupled by the particular airframe.

As specified, these desired dynamics are primarily generated by the elevator deflection, δ_e In view of this, we decided to place the desired dynamics in the first column of $D'(s)$ to indicate the major elevator activity. Meanwhile the second column of $D'(s)$ should have insignificant responses. Therefore, we let α/c and q/c be the (1,1) and (2,1) element of $D'(s)$, respectively. Since we want the responses of the second column elements of $D'(s)$ to be insignificant and keeping in view the degree constraints for 'good fit', we take the transfer function for each of the (1,2) and (2,2) elements in $D'(s)$ as $1/(s+50)^2$. Thus,

$$\mathbf{D}'(s) = \begin{bmatrix} \dfrac{21.2}{s^2 + 7s + 25} & \dfrac{1}{(s + 50)^2} \\ \dfrac{21.1(s + 1.08)}{s^2 + 7s + 25} & \dfrac{1}{(s + 50)^2} \end{bmatrix} \tag{4.32}$$

In order to satisfy the degree condition as in remark 5 in Chapter 3, we enhance D′(s) by multiplying the (2,1) element by 50/(s+50). Thus, the 'desired' closed loop transfer function matrix is taken as

$$\mathbf{D}(s) = \frac{\begin{bmatrix} 21.2s^2 + 2120s + 53000 & s^2 + 7s + 25 \\ 1055s^2 + 53889.4s + 56970 & s^2 + 7s + 25 \end{bmatrix}}{s^4 + 107s^3 + 3225s^2 + 20000s + 62500} \tag{4.33}$$

4.2.6 The Design

In this example, the matching of the first column elements of D(s) is of paramount importance. The bandwidth of the (1,1) element is found to be [0,5] rad/sec, and of the (2,1) element is found to be [2.5,8] rad/sec. Therefore the frequency interval of [0,8] or less should be considered for the minimization of the error function E_m. Using the program with the plant in Equation (4.28) and the 'desired' transfer function matrix in Equation (4.33), we varied the frequency interval and the order of the controller until we obtained satisfactory results. Fairly satisfactory results were obtained by using the frequency interval, [0,2] for $1 \le i, k \le 2$, with a second order controller. The proposed technique yielded the following transfer function:

$$\mathbf{C}(s) = \frac{\begin{bmatrix} c_{11}(s) & c_{12}(s) \\ c_{21}(s) & c_{22}(s) \end{bmatrix}}{c(s)} \tag{4.34}$$

where

$$
\begin{aligned}
c_{11}(s) &= -0.14952s^2 + 19.85508s + 19.15416 \\
&= -0.14952(s + 0.95779)(s - 133.75) \\
c_{12}(s) &= -0.0010493s^2 + 0.0099626s + 0.008956 \\
&= -0.0010493(s + 0.82694)(s - 10.321) \\
c_{21}(s) &= 2.23717s^2 + 1.31541s + 1.79723 \\
&= 2.23717(s^2 + 0.58798s + 0.80335) \\
c_{22}(s) &= 0.022089s^2 + 0.0003915s + 0.003185 \\
&= 0.022089(s^2 + 0.017727s + 0.1442) \\
c(s) &= s^2 + 6.6078s + 3.52986 \\
&= (s + 0.5862)(s + 6.0216)
\end{aligned}
$$

Note that the controller C(s) is stable. Also the compensated closed loop transfer function matrix is given by

$$
\mathbf{F}(s) = [\mathbf{I}_2 + \mathbf{G}(s)\mathbf{C}(s)\mathbf{H}(s)]^{-1}\mathbf{G}(s)\mathbf{C}(s) \tag{4.35}
$$

$$
= \frac{\begin{bmatrix} f_{11}(s) & f_{12}(s) \\ f_{21}(s) & f_{22}(s) \end{bmatrix}}{f(s)} \tag{4.36}
$$

where

$$
\begin{aligned}
f_{11}(s) &= 3.29517(s^2 + 0.016392s + 0.0029554)(s^2 + 0.016302s + 0.02247) \\
&\quad \times (s + 0.58597)(s + 0.98225)(s + 6.0649)(s + 10.728)(s^2 + 44.29s + 9988.9)
\end{aligned}
$$

$$f_{12}(s) = 0.036783(s^2 + 0.016365s + 0.0028624)(s^2 + 0.016619s + 0.022267)$$
$$\times(s + 0.5862)(s^2 + 2.9794s + 2.6362)(s + 6.0216)(s^2 + 13.049s + 1754.3)$$
$$f_{21}(s) = -62.302s(s + 0.025953)(s^2 + 0.016325s + 0.022445)(s + 0.5862)$$
$$\times(s + 0.98307)(s + 1.059)(s + 6.0216)(s + 13)(s - 440.03)$$
$$f_{22}(s) = -0.029687s(s + 0.029317)(s^2 + 0.017013s + 0.022136)(s + 0.29824)$$
$$\times(s + 0.97521)(s^2 + 1.4627s + 6.5752)(s + 18.807)(s - 476.78)$$
$$f(s) = (s^2 + 0.017205s + 0.0024661)(s^2 + 0.016417s + 0.022829)(s + 0.58597)$$
$$\times(s + 0.98263)(s^2 + 7.0979s + 24.672)(s + 6.0645)(s + 12.471)$$
$$\times(s^2 + 14.138s + 1353.2)$$

4.2.7 Design Comparison

The frequency responses of each element of the compensated and the 'desired' closed loop transfer function matrices are given in Figures 4.25 through 4.32. Since the magnitude responses of the compensated and the 'desired' system of the (1,2) and (2,2) elements are relatively very small as shown in Figure 4.26 and Figure 4.28, therefore the matching of the phase responses of the elements are not of interest. And also, the time responses due to the unit step inputs $\begin{bmatrix} 1^\circ \\ 0 \end{bmatrix}$ of the compensated and the 'desired' systems are given in Figures 4.33 and 4.34. Now the transfer function matrix of the elevator and flap deflections of the airframe with respect to the pilot commanded input is given by

$$\mathbf{L}(s) = \frac{\begin{bmatrix} l_{11}(s) & l_{12}(s) \\ l_{21}(s) & l_{22}(s) \end{bmatrix}}{l(s)} \tag{4.37}$$

where

$$
\begin{aligned}
l_{11}(s) &= -9.7188(s^2 + 0.016522s + 0.0030627)(s^2 - 0.01626s - 0.022566) \\
&\quad \times (s + 0.58561)(s + 0.98206)(s - 3.1956)(s + 5.7109)(s - 126.39) \\
&\quad \times (s^2 + 11.846s + 49.841) \\
l_{12}(s) &= -0.068207(s^2 + 0.016975s + 0.0027477)(s^2 + 0.01626 + 0.022566) \\
&\quad \times (s + 0.93667)(s^2 + 0.21863s + 0.26736)(s^2 + 7.4103s + 26.911) \\
&\quad \times (s^2 + 3.6697s + 1179.2) \\
l_{21}(s) &= 29.0836s(s + 0.026042)(s^2 + 0.014474s + 0.024529)(s + 0.57672) \\
&\quad \times (s + 0.97889)(s^2 + 0.59512s + 0.83605)(s + 5.8277) \\
&\quad \times (s^2 + 14.335s + 1376.4) \\
l_{22}(s) &= 0.2872(s^2 + 0.01738s + 0.0023487)(s^2 + 0.016484s + 0.023031) \\
&\quad \times (s + 0.9656\text{[illegible]})(s^2 - 0.00002985s + 0.11558)(s^2 + 6.6353s + 23.155) \\
&\quad \times (s^2 + 14.149s + 1354.5) \\
l(s) &= (s^2 + 0.017205s + 0.0024661)(s^2 + 0.016417s + 0.022829)(s + 0.58597) \\
&\quad \times (s + 0.98263)(s^2 + 7.0979s + 24.672)(s + 6.0645)(s + 12.471) \\
&\quad \times (s^2 + 14.138s + 1353.2)
\end{aligned}
$$

The time response of the elevator and flap deflections of the airframe due to the unit step command input $\begin{bmatrix} 1^\circ \\ 0 \end{bmatrix}$ of the compensated system are also given in Figures 4.35 and 4.36. These figures, indeed, bring out the expected results that the elevator deflection is the major control surface for the desired dynamics of the aircraft.

Comparison of the magnitude, phase and time responses of the compensated system with C(s) with the 'desired' system as shown in Figures 4.25 through 4.34 shows that the results obtained from the proposed design technique are excellent.

4.3 YF–16 CCV. Case II

We pursue the same aircraft example viz. YF–16 CCV whose model and actuator for longitudinal control are given in (4.2). In this case, Mr. Tom Gentry, the Project Engineer, gave us the following 'desired' dynamics:

$$\frac{\alpha}{c} = \frac{13.1}{s^2+7.7s+30.25}\ \frac{\text{rad}}{\text{unit}} \tag{4.38}$$

$$\frac{q}{c} = \frac{13.0(s+1.25)}{s^2+7.7s+30.25}\ \frac{\text{rad/sec}}{\text{unit}} \tag{4.39}$$

The justification for these desired dynamics is similar to the one in (4.2) and we don't want to dwell on that. For reasons similar to the ones for (4.2), we keep the desired dynamics in the first column of the desired transfer function matrix. Keeping in view the degree condition (remark 5, Chapter 3) as elaborated in (4.2), we select the following transfer function matrix D(s) as the 'desired' closed loop system:

$$D(s) = \begin{bmatrix} \dfrac{13.1}{s^2+7.7s+30.25} & \dfrac{1}{(s+50)^2} \\ \dfrac{13.0(s+1.25)}{s^2+7.7s+30.25}\cdot\dfrac{50}{s+50} & \dfrac{1}{(s+50)^2} \end{bmatrix} \tag{4.40}$$

After a few 'trial and error' attempts, we settled on a second order controller with [0,4] as the interval of integration for all components in the error function. The proposed technique yielded the following transfer function matrix for the controller C(s):

$$C(s) = \frac{\begin{bmatrix} c_{11}(s) & c_{12}(s) \\ c_{21}(s) & c_{22}(s) \end{bmatrix}}{c(s)} \tag{4.41}$$

where

$$
\begin{aligned}
c_{11}(s) &= 0.0020384s^2 + 0.69268s + 0.97256 \\
&= 0.0020384\,(s+1.4099)(s+338.41) \\
c_{12}(s) &= -0.000026447s^2 + 0.00039187s + 0.0012705 \\
&= -0.000026447(s+2.7367)(s-17.554) \\
c_{21}(s) &= 0.07805s^2 - 0.0032581s - 0.87825 \\
&= 0.078085\,(s+3.1516)(s-3.5688) \\
c_{22}(s) &= -0.00033998s^2 + 0.00012675s - 0.025865 \\
& -0.00033998\,(s-0.18641 \pm 8.7203j) \\
c(s) &= 0.061069s^2 + 0.41695s + 1 \\
&= 0.061069\,(s+3.4137 \pm 2.1729j)
\end{aligned}
\tag{4.42}
$$

Both C(s) and the closed loop transfer function matrix F(s) are stable for this design.

The frequency responses of each element of the compensated and the 'desired' closed loop transfer function matrices are given in figures 4.37 through 4.44. Since the magnitude responses of the (1,2) and (2,2) elements in the 'desired' transfer function matrix were almost zero and the corresponding responses in the compensated system turn out to be small (Figures 4.38 and 4.40), no attention or significance is attached to the phase-mismatch for the (1,2) and (2,2) elements. However, the match for the magnitude and phase responses for (1,l) and (2,l) elements (Figures 4.33, 4.39, 4.41, 4.43) is excellent showing thereby the second order controller obtained by the proposed technique gives excellent results. The time responses due to unit step input $\begin{bmatrix} 1^\circ \\ 0 \end{bmatrix}$ of the compensated and the 'desired' systems are shown in Figures 4.45 and 4.46. Again good results are confirmed by the time response.

4.4 YF–16 CCV. Case III

For the same aircraft model studied in (4.2) and (4.3), we design a control system by using the proposed technique to satisfy another set of specifications given to us by Mr. Tom Gentry, the Project Engineer.

The desired dynamics are:

$$\frac{q}{c} = \frac{1}{s^2+7s+25} \tag{4.43}$$

$$\frac{\gamma}{c} \approx 0 \tag{4.44}$$

where γ is the flight path angle. With q and γ as outputs, and δ_e, the horizontal tail deflection and δ_f, the flap deflection as inputs, the transfer function matrix of the plant for the aircraft of (4.2) is derived as

$$P(s) = \frac{\begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}}{P(s)} \tag{4.45}$$

where

$$\begin{aligned} P_{11}(s) &= 274.25s^3 + 303.54s^2 + 7.6929s \\ P_{12}(s) &= 63.797s^3 + 90.402s^2 + 2.0213s \\ P_{21}(s) &= 1.9898s^3 + 1.7819s^2 - 299s - 6.8676 \\ P_{22}(s) &= 2.1378s^3 + 2.3051s^2 - 89.322s - 1.8278 \\ P(s) &= s^5 + 15.138s^4 + 17.174s^3 - 138.3s^2 - 2.1981s - 0.4493 \end{aligned} \tag{4.46}$$

We, first, design a stabilizing controller T(s) as shown in Figure 4.47 and given by

$$T(s) = \frac{5(s^2 + 0.01626s + 0.022566)}{s(s + 0.02)} \tag{4.47}$$

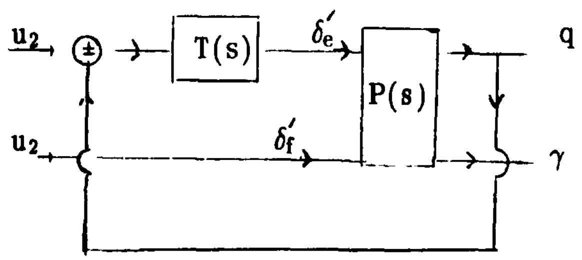

Figure 4.47: Stabilization of the Plant

With T(s) in the loop as in Figure 4.47 the closed loop transfer function matrix G(x) is calculated as

$$G(s) = \frac{\begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & q_{22}(s) \end{bmatrix}}{g(s)} \tag{4.48}$$

where

$$\begin{aligned}
q_{11}(s) &- 1371.2s^6 + 19366s^5 + 20114s^4 + 1258s^3 + 454.23s^2 + 11.284s \\
g_{12}(s) &= 63.797s^6 + 921.05s^5 + 1195.6s^4 + 49.822s^3 + .052553s^3 \\
g_{21}(s) &= 9.9489s^6 + 138.41s^5 - 1376.7s^4 - 19488s^3 - 794.08s^2 - 446.6s - 10.073 \qquad (4.49) \\
g_{22}(s) &= 2.1378s^6 + 30.14s^5 + 2238s^4 - 1100.8s^3 + 5.2304s^2 + 0.11178s \\
g(s) &= s^8 + 28.158s^7 + 1585.8s^6 + 19455s^5 + 18316s^4 + 1193s^3 + 447.8s^2 + 11.167s
\end{aligned}$$

Keeping the degree condition (remark 5, chapter 3) in mind, the 'desired' transfer function matrix is taken as

$$D(s) = \begin{bmatrix} \frac{1}{s^2+7s+25} & \frac{1}{(s+1000)^2} \\ \frac{1}{(s+1000)^2} & \frac{1}{(s+1000)^2} \end{bmatrix} \tag{4.50}$$

By using the proposed technique, a second order controller was designed such that the controller and the compensated closed loop system are stable and the frequency responses of the compensated system and the 'desired' system are close. The controller transfer function matrix C(s) by the proposed technique is given by

$$C(s) = \frac{\begin{bmatrix} c_{11}(s) & c_{12}(s) \\ c_{21}(s) & c_{22}(s) \end{bmatrix}}{c(s)} \tag{4.51}$$

where

$$\begin{aligned}
c_{11}(s) &= 0.000072834\,(s + 5.1914 \pm 23.592j) \\
c_{12}(s) &= -0.000035781\,(s + 0.14002 \pm 9.2895j) \\
c_{21}(s) &= -0.00096978\,(s + 3.4304 \pm 5.8456j) \\
c(s) &= 0.040639\,(s + 3.5046 \pm 3.5106j) \qquad (4.52)
\end{aligned}$$

The magnitude and phase responses of the compensated and the 'desired' transfer function matrices are shown in Figures 4.48 through 4.55. Except for not so good a fit at very low frequency in Figure 4.48, these plots clearly demonstrate that the design carried out by the proposed technique gives good results.

Summary

Two numerical illustrative examples have been studied in this chapter which show that the design carried out by the proposed technique gives excellent results.

5. ROBUSTNESS ANALYSIS FOR YF–16 CCV

In this chapter, we carry out a modest robustness analysis for the aircraft of example 2. The best way to approach this would be to vary several parameters in the model of the plant and then analyze the stability and performance robustness of the design. But this is quite involved and quite time consuming. We thus settled for a less lofty goal. We decided to vary one parameter in the characteristic polynomial of the plant after it was stabilized.

The marginally stabilized plant for the F–16 CCV for example is given by Equation (4.28). The characteristic polynomial g(s) is given by

$$g(s) = s^8 + 28.158s^7 + 1585.8s^6 + 19455s^5 + 18316s^4 + 1193s^3 + 447.8s^2 + 11.167s \qquad (5.1)$$

We decided to study the stability and performance robustness of the compensated system by varying the coefficient of s^7 in (5.1). We denote this coefficient by δ. Thus the nominal value of δ is 28.158. We found that for δ in the interval [20,34], the inner loop of the control system in Figure 4.3 stays marginally stable. With the controller C(s) designed for the nominal plant in Example 2 and given by Equation (4.34), the overall closed loop system represented by Figure 4.24 also remains stable. Thus for δ in [20,34] both the inner loop and the overall system remain stable, thereby establishing the stability robustness of the design over this range of δ. These results have been established by calculating the poles of the system by varying δ over [20,34] with a fairly small step size.

Now the performance robustness. Figure 5.1 shows that the graph of the error function between the frequency responses of the compensated closed loop system and the desired closed loop system over the frequency interval [0,10] rad/sec when δ is varied over the interval [20,34]. The value of the error for the nominal value of δ(=28.158) is 0.375. From Figure 5.1 it is clear that the error function for other values of δ in [20,34] remains

between 0 and 1. This establishes performance robustness of the design when only δ is varied over [20,34]. To reinforce the performance robustness of the design, we have drawn the magnitude and the phase responses of the compensated and the 'desired' transfer function matrices for some sample values of δ in [20,34]. In fact, figures are drawn for $\delta =$ 20, 25, 30, and 34. These figures also portray a good performance robustness of the design when δ is varied. For these values of δ, the magnitude and phase responses are shown in Figures 5.2 through 5.33. Unit step responses are shown in Figures 5.34 through 5.41.

As was mentioned earlier, this stability and performance robustness analysis is only a small part of the overall robustness analysis of the system. But the results given by the analysis given in this report are fairly encouraging and make one optimistic about the robustness of the design by the proposed technique.

A complete robustness analysis of the proposed technique is left for future research work.

Summary

In this chapter, a minor robustness analysis for stability and performance for the aircraft example has been carried out. It is observed that the design carried out by the frequency matching technique of this project gives fairly decent robustness results when only one parameter in the plant is varied.

BIBLIOGRAPHY

[1] Chen, C.F., "Model Reduction of Multivariable Control Systems by Means of Matrix Continued Fraction," *Int. J. Control*, 1974, **20**, pp. 225–238.

[2] Chen, C.F. and Shieh, L.S., "An Algebraic Method for Control System Design," *Int. J. Control*, 1970, **11**, pp. 717–739.

[3] Doyle, J.C. and Stein, G., "Multivariable Feedback Design: Concepts for Classical/Modern Synthesis," *IEEE Trans. Automat. Control*, 1981, **AC–26**, pp. 4–16.

[4] Gentry, T.A. and Pujara, L.R., "An Example of Preliminary Longitudinal Flying Qualities Design Using a Frequency Matching Method," *Proc. of IEEE NAECON*, 1987, pp. 523–529.

[5] Gupta, N.K., "Frequency–Shaped Cost Functionals: Extensions of Linear-Quadratic–Gaussian Design Methods," *J. of Guidance, Control and Dynamics*, November–December 1980, **6**, pp. 529–535.

[6] Hiroyuki, I. and Makoto, K., "Disturbance Attenuation by a Frequency–Shaped Linear–Quadratic–Regulator Method," *J. of Guidance, Control and Dynamics*, July–August 1986, **9**, pp. 397–402.

[7] Hoh, Roger H., et al., "Development of Handling Quality Criteria for Aircraft with Independent Control of Six Degrees of Freedom," AFWAL–TR–81–3027, April 1981.

[8] Hoh, Roger H., et al., "Proposed MIL Standard and Handbook – Flying Qualities of Air Vehicles. Vol. I: Proposed MIL Standard. Vol. II: Proposed MIL Handbook," AFWAL–TR–82–3081, November 1982.

[9] Lehtomaki, N., Stein, G. and Walls, J., "Multivariable Prefilter Design for Command Shaping," *AIAA Conference*, 1984, pp. 38–43.

[10] Levy, E.C., "Complex Curve Fitting", *IRE Trans.*, 1959, **AC–4**, pp. 37–43.

[11] McFarlane, A.G.J. and Postlethwaite, I., "The Generalized Nyquist Stability Criterion and Multivariable Root–Loci," *Int. J. Control*, 1977, **23**, pp. 81–128.

[12] Moorhouse, D.J. and Woodcock, R.J., "Background Information and User–Guide for MIL–F–8785C, Military Specifications Flying Qualities of Piloted Airplanes," AFWAL–TR–81–3109, WPAFB, Ohio, July 1982.

[13] Pujara, L.R., "Computer–Aided Control System Design Technique with Applications to Aircraft Flying Qualities," *J. of Guidance, Control and Dynamics*, 1988, **11**, pp. 250–255.

[14] Pujara, L.R., "A Note on Model Reduction of Control Systems," *J. of Guidance, Control and Dynamics*, 1985, **8**, pp. 152–155.

[15] Pujara, L.R. and Rattan, K.S., "A Frequency Matching Method for Model Reduction of Digital Control Systems," *Int. J. Control*, 1982, **35**, pp. 139–148.

[16] Pujara, L.R. and Rattan, K.S., "Model Reduction of Linear Multivariable Digital Control Systems via Frequency Matching," *IFAC Symposium on Computer–Aided Design of Multivariable Technological Systems*, 1982, pp. 561–565.

[17] Pujara, L.R. and Rattan, K.S., "Model Reduction of Linear Multivariable Control Systems via Frequency Matching," *IEE Proc. of Control Theory and Application*, 1984, **131**, pp. 243–247.

[18] Rattan, K.S., "Digitalization of Existing Continuous Control Systems," *IEEE Trans. Automat. Control*, 1984, **AC–29**, pp. 282–285.

[19] Rattan, K.S. and Yeh, H.H., "Discretizing Continuous Data Control Systems," *Comput. Aided Des.*, 1978, **10**, pp. 299–306.

[20] Rao, S.V. and Lamba, S.S., "A New Frequency Domain Technique for Simplification of Linear Dynamic Systems," *Int. J. Control*, 1974, **20**, pp. 71–79.

[21] "Military Specification–Flying Qualities of Piloted Airplanes," MIL–F–8785C, 5 November 1980.

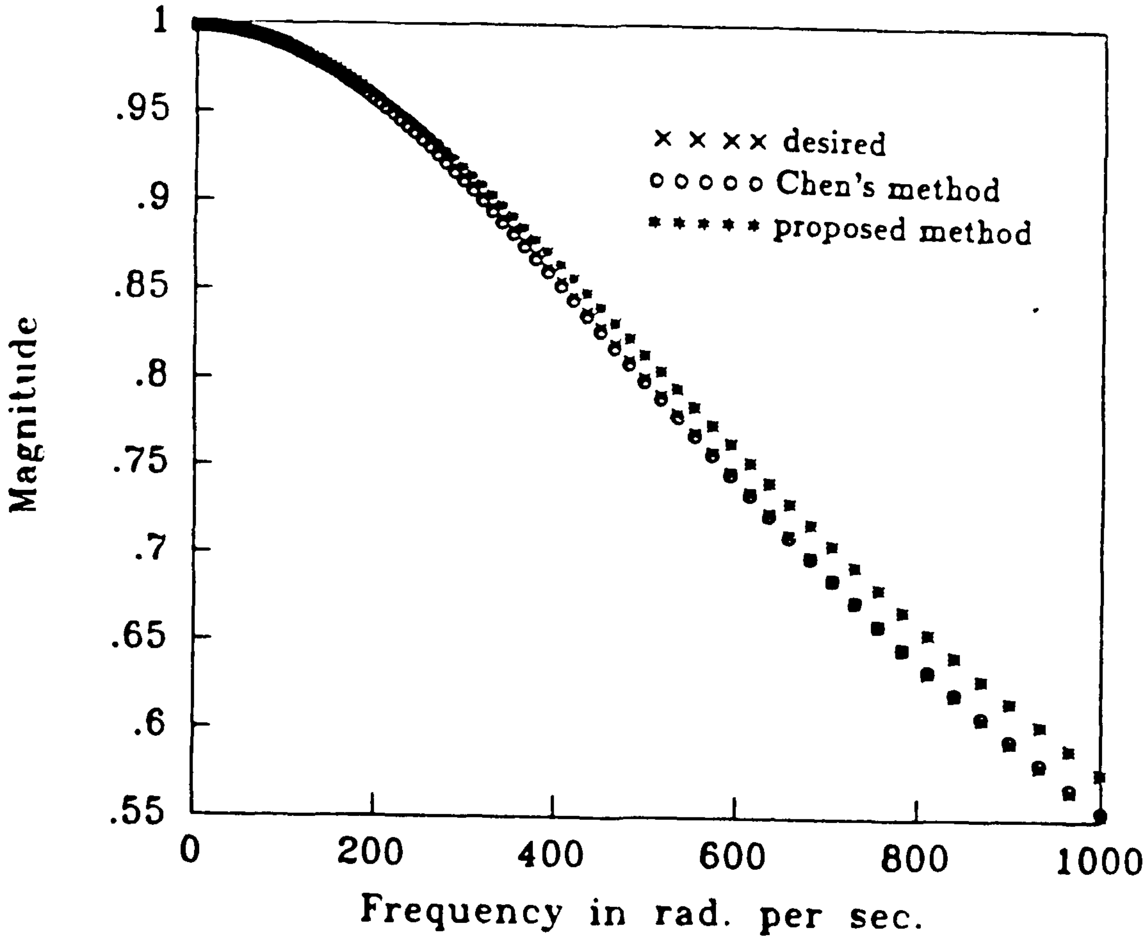

Figure 4.1: Linear magnitude response of the (1,1) element of the closed loop transfer function matrix with zero order controller for Example 1

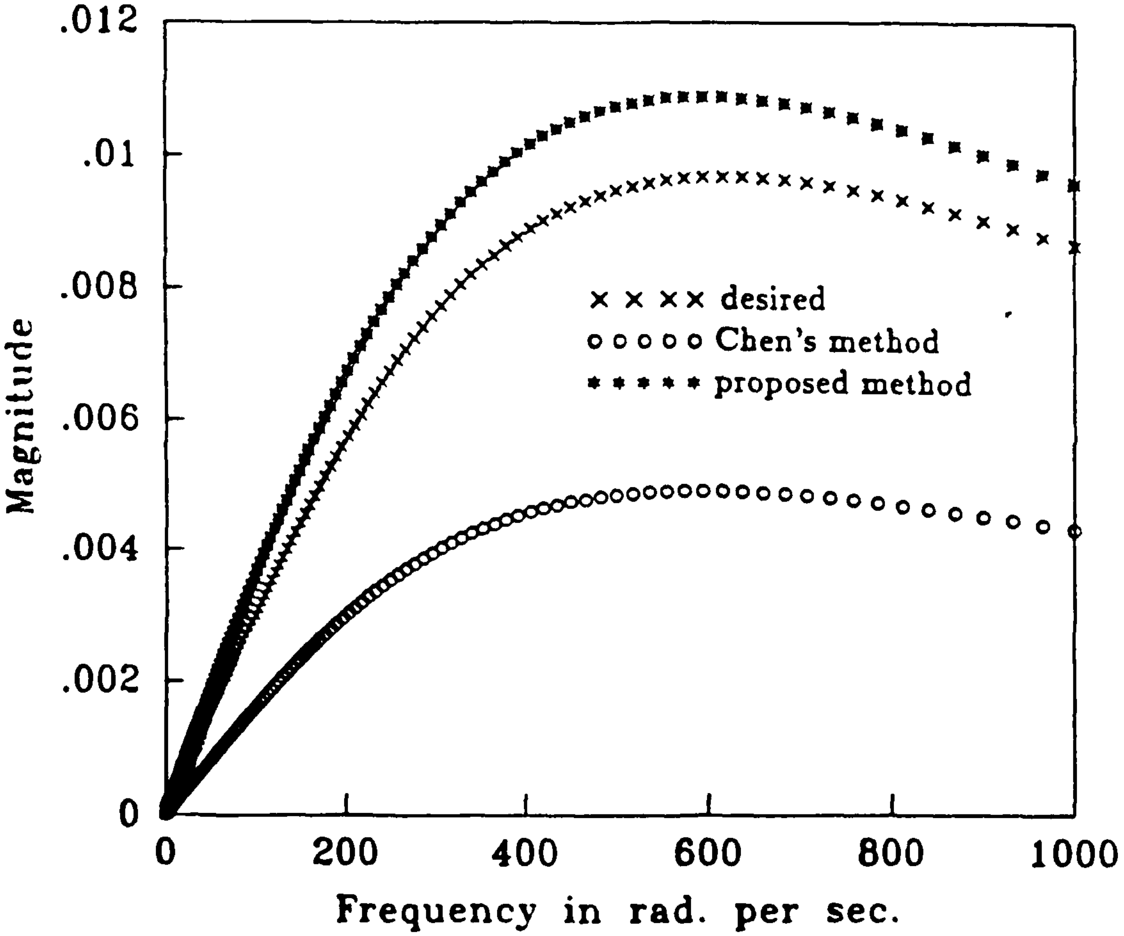

Figure 4.2: Linear magnitude response of the (1,2) element of the closed loop transfer function matrix with zero order controller for Example 1

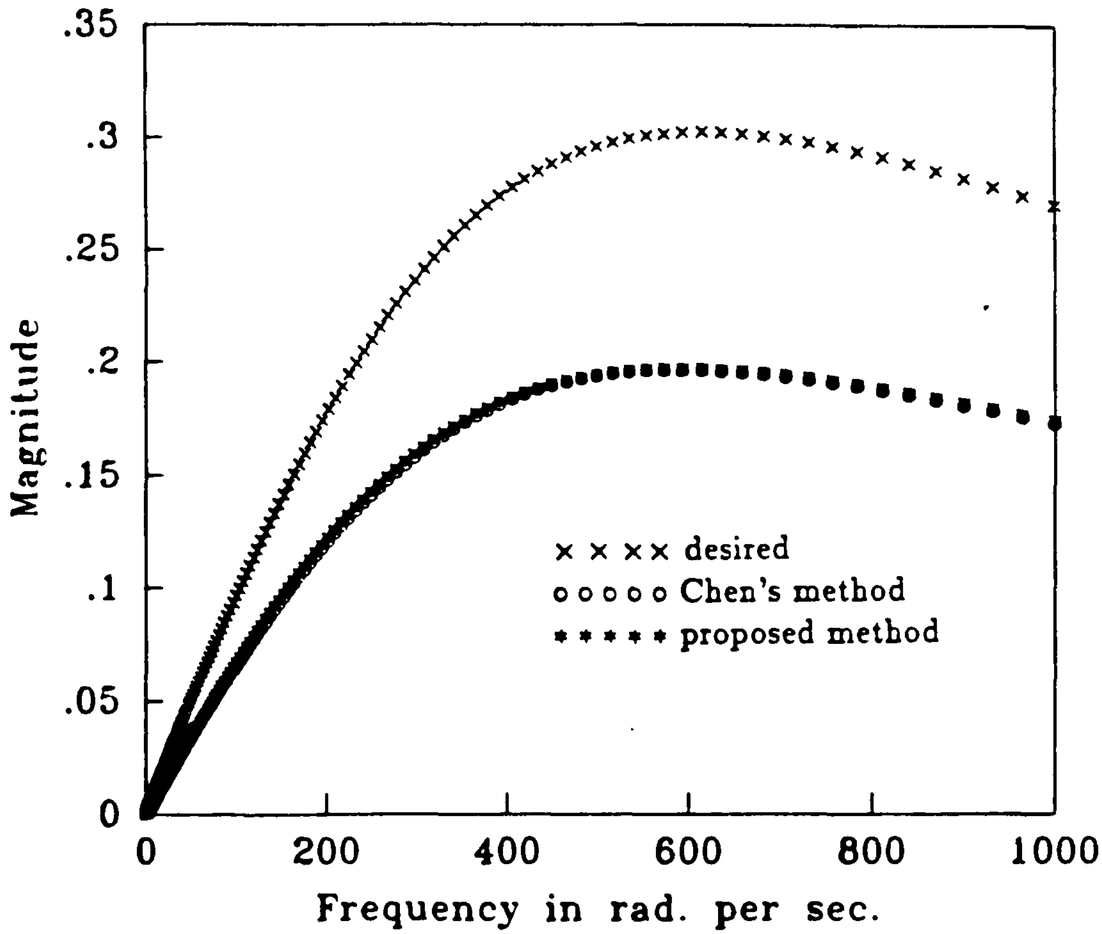

Figure 4.3: Linear magnitude response of the (2,1) element of the closed loop transfer function matrix with zero order controller for Example 1

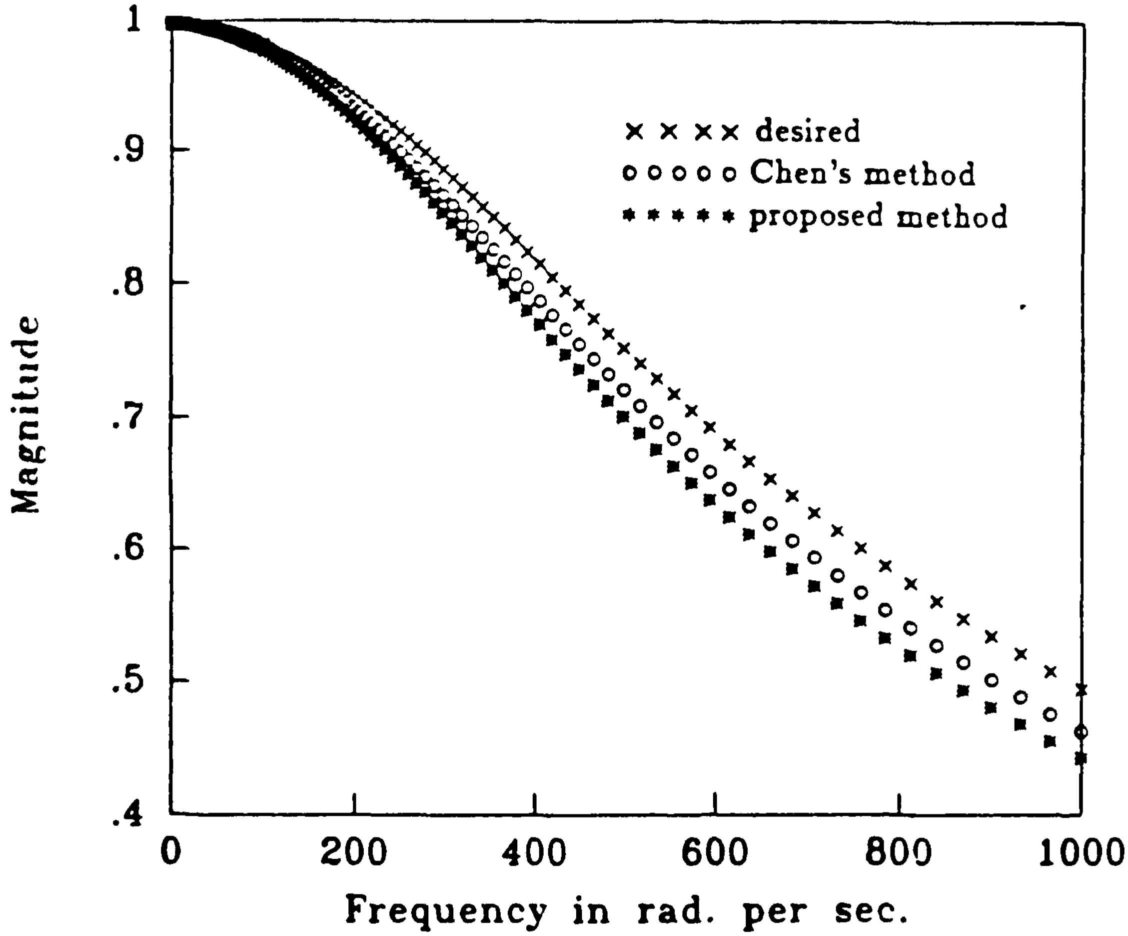

Figure 4.4: Linear magnitude response of the (2,2) element of the closed loop transfer function matrix with zero order controller for Example 1

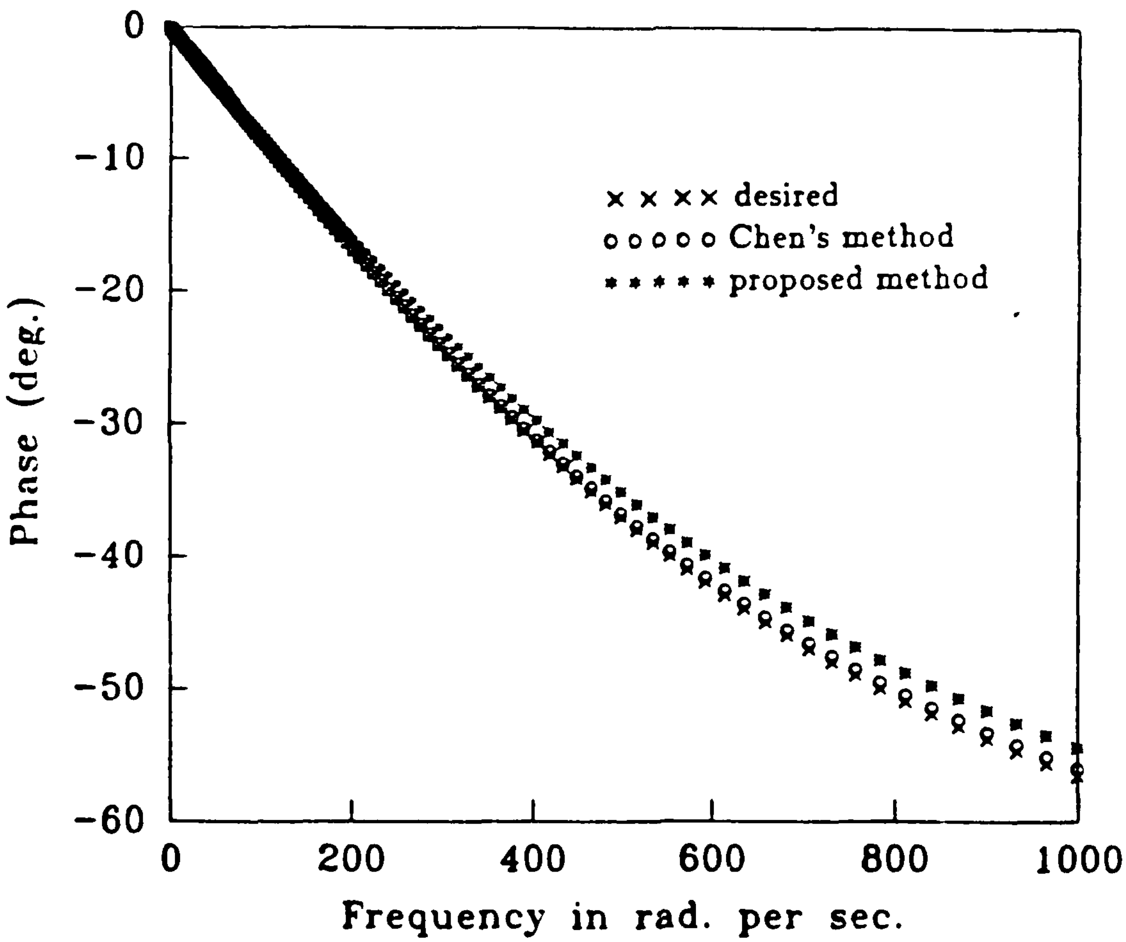

Figure 4.5: **Phase response of the (1,1) element of the closed loop transfer function matrix with zero order controller for Example 1**

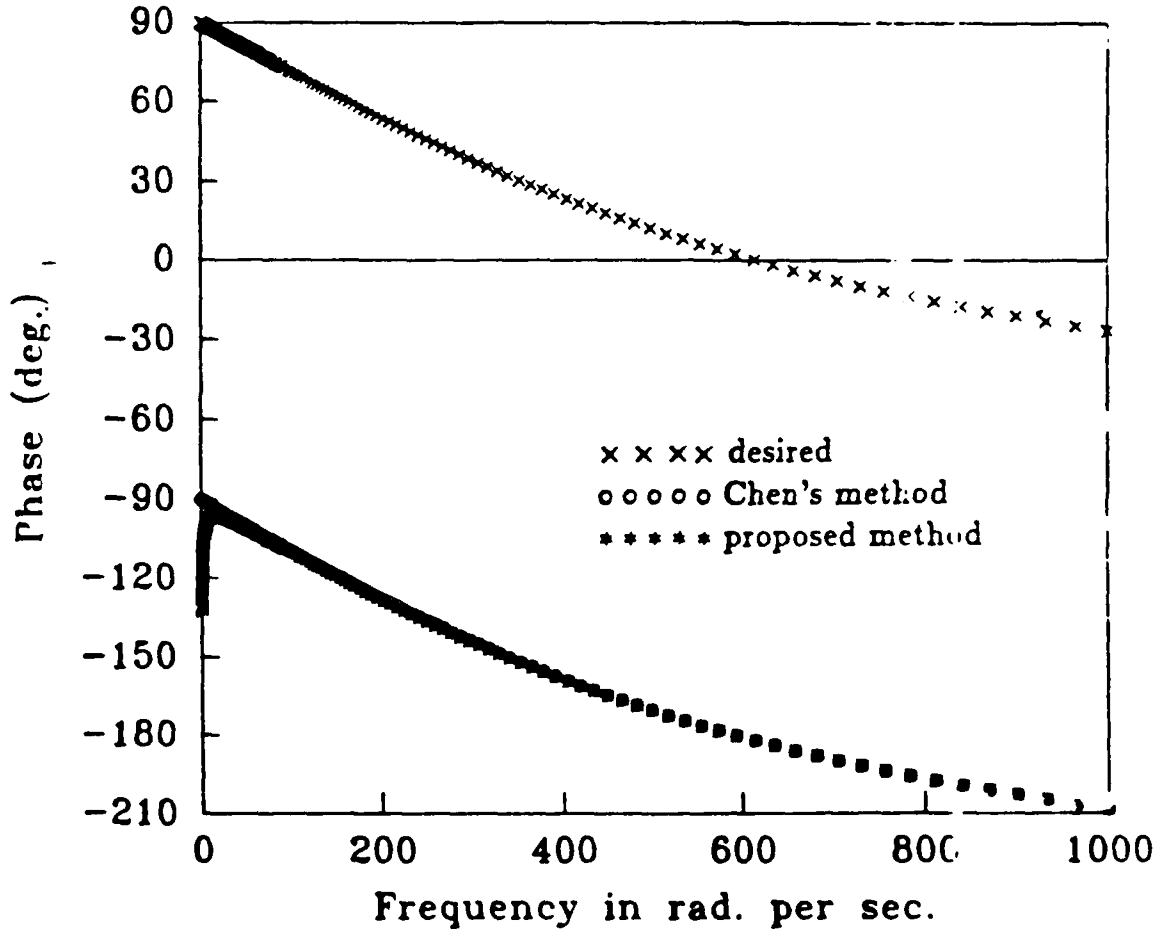

Figure 4.6: Phase response of the (1,2) element of the closed loop transfer function matrix with zero order controller for Example 1

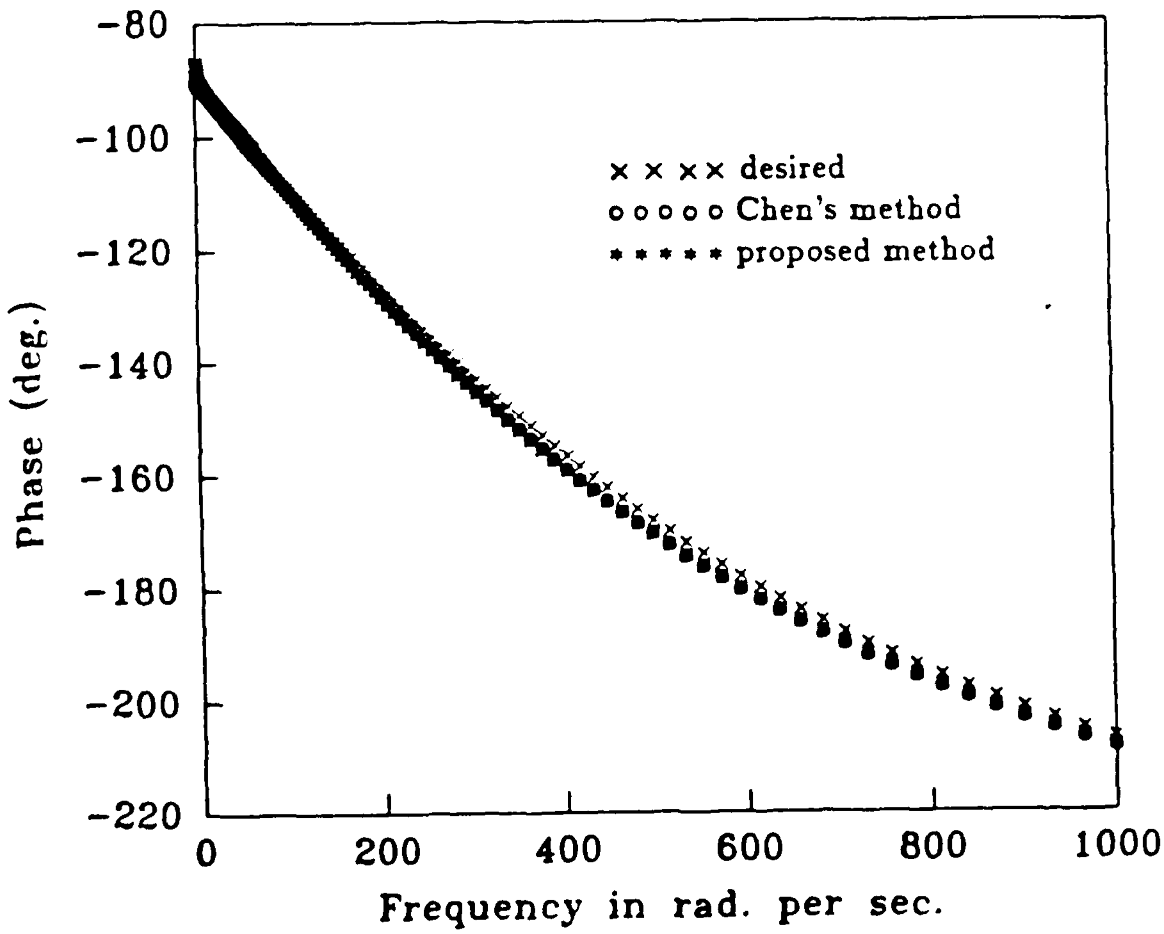

Figure 4.7: Phase response of the (2,1) element of the closed loop transfer function matrix with zero order controller for Example 1

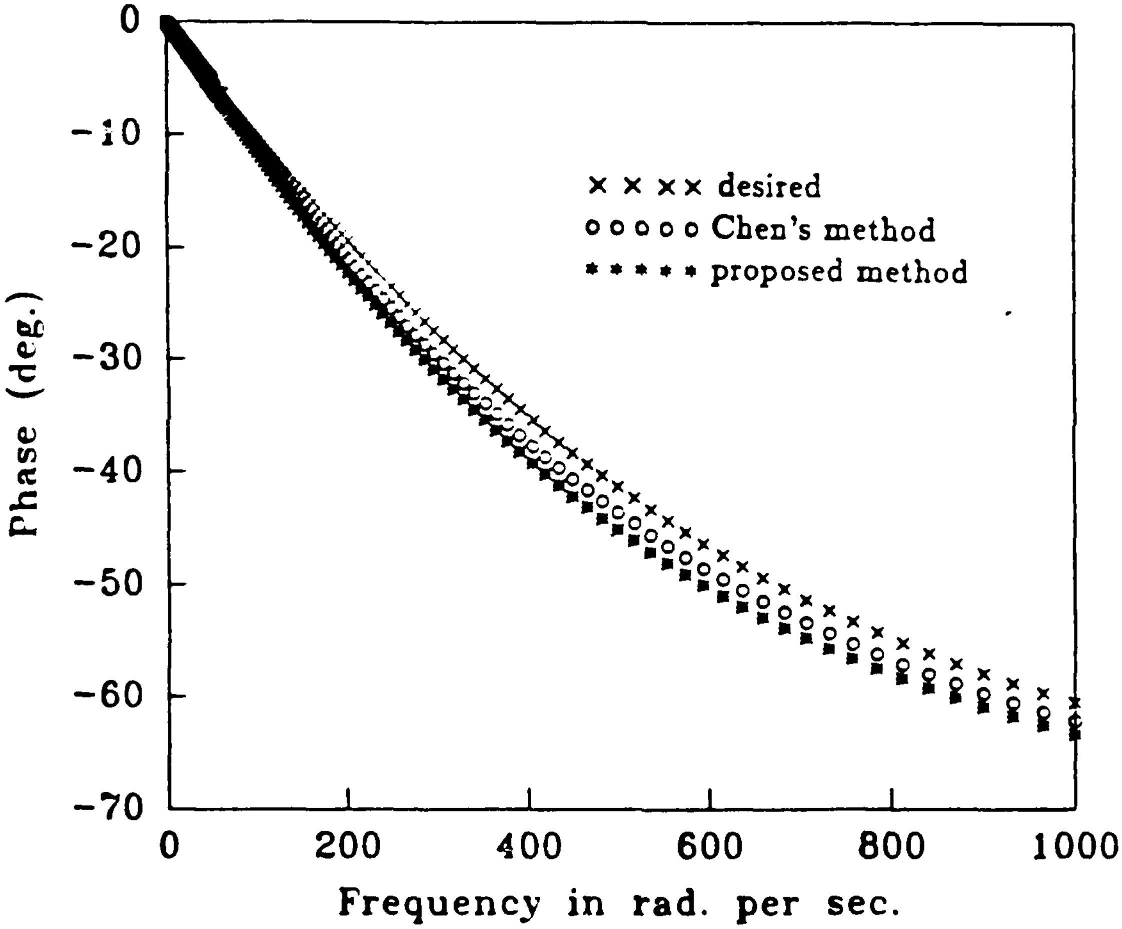

Figure 4.8: Phase response of the (2,2) element of the closed loop transfer function matrix with zero order controller for Example 1

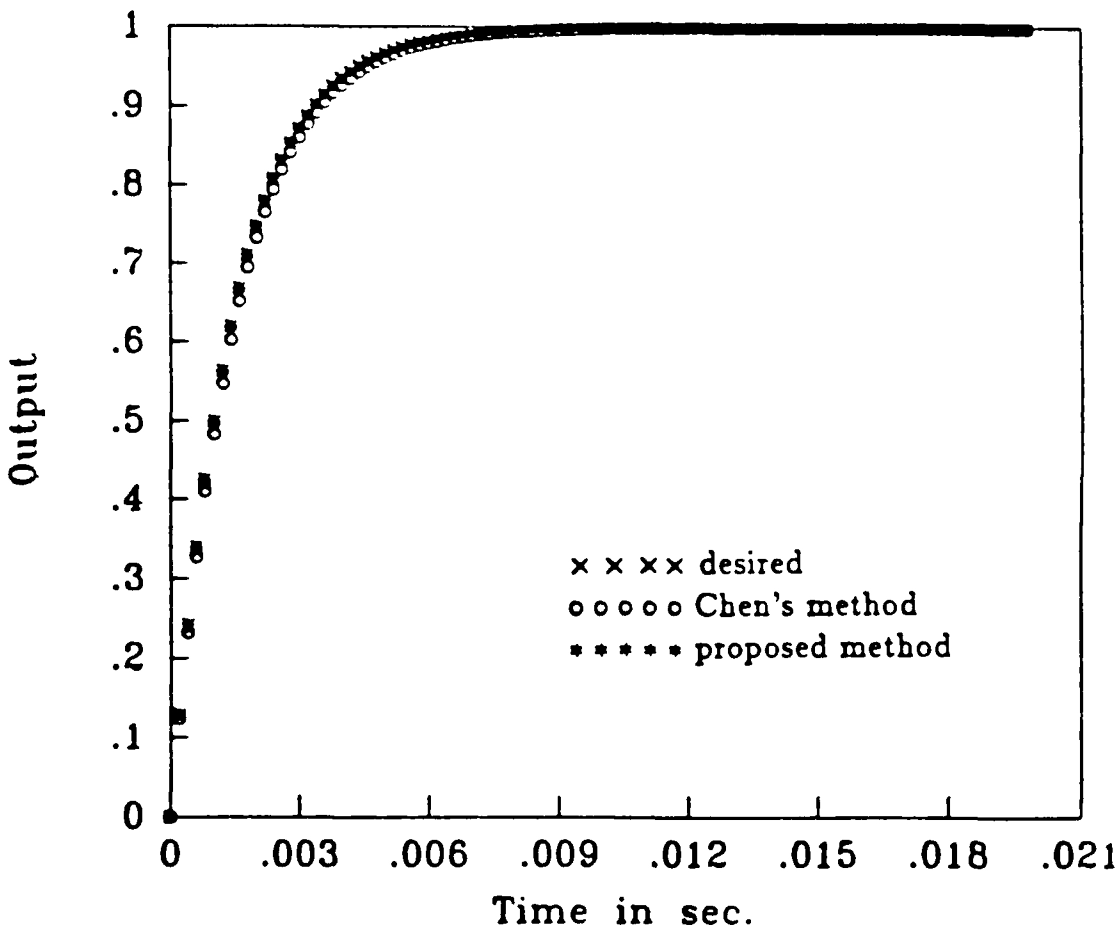

Figure 4.9: Time response of the first output of the closed loop system with zero order controller for Example 1

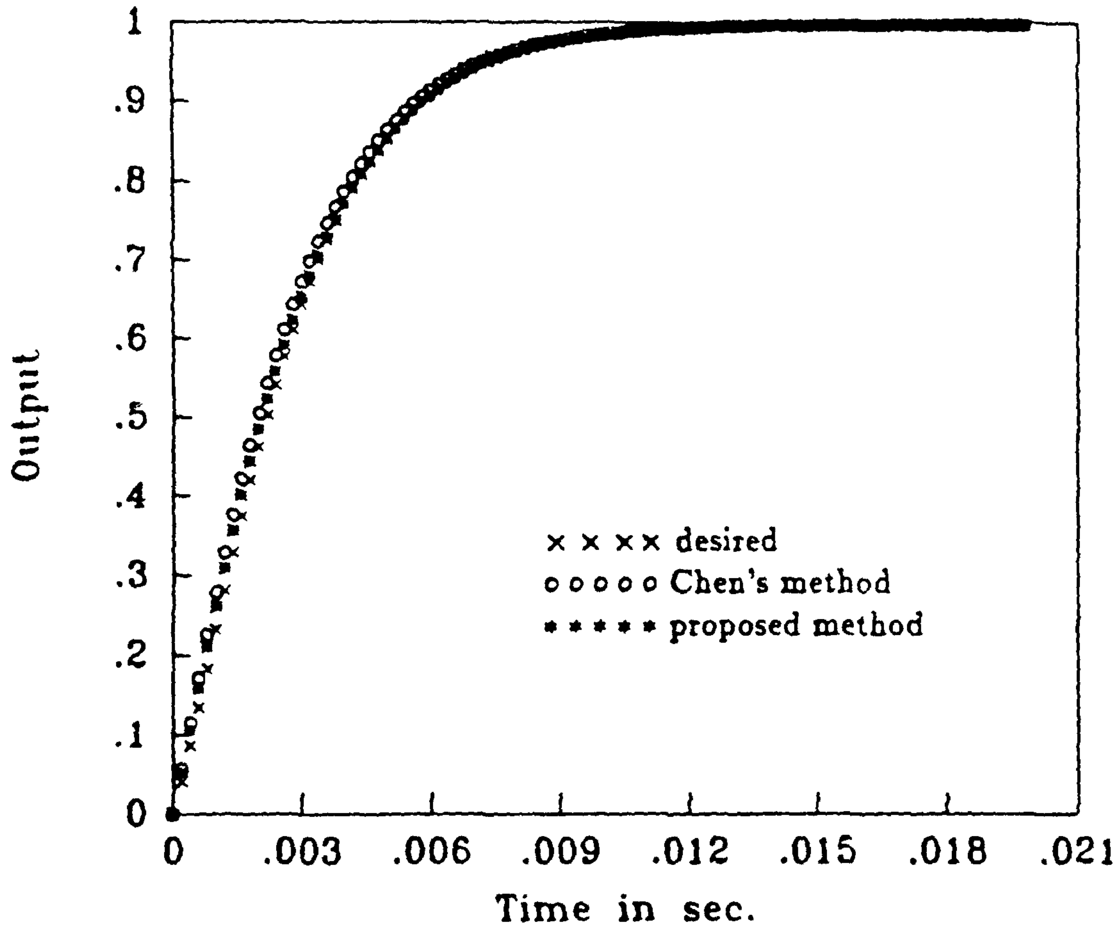

Figure 4.10: Time response of the second output of the closed loop system with zero order controller for Example 1

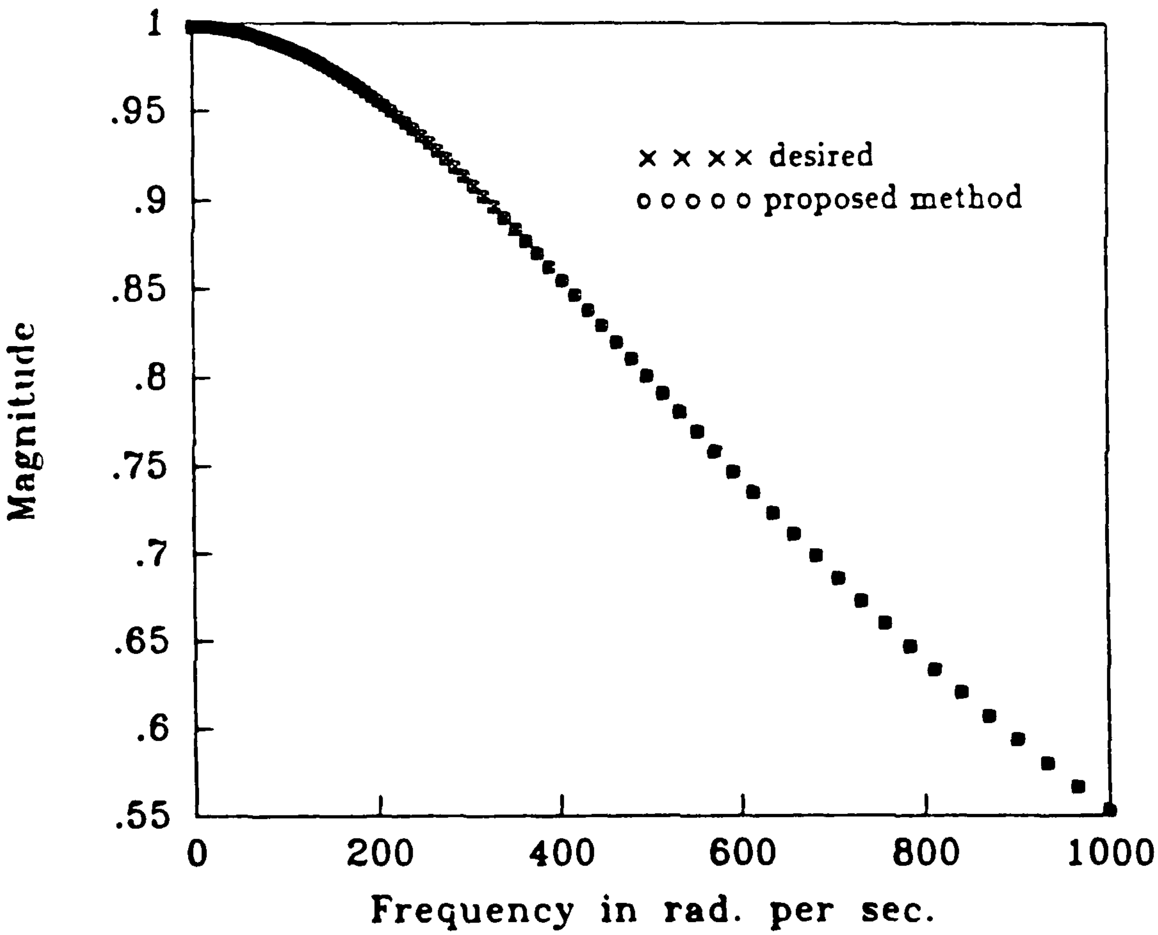

Figure 4.11: Linear magnitude response of the (1,1) element of the closed loop transfer function matrix with first order controller for Example 1

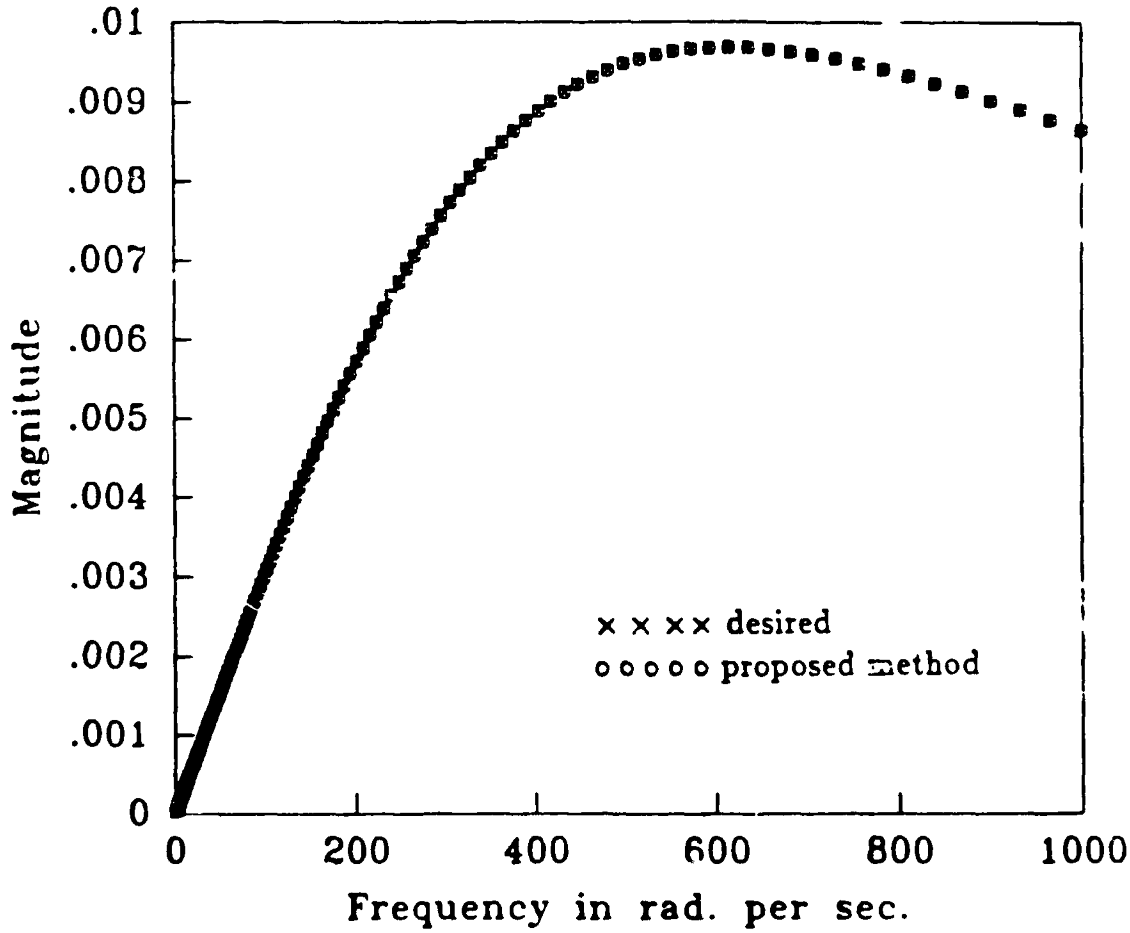

Figure 4.12: Linear magnitude response of the (1,2) element of the closed loop transfer function matrix with first order controller for Example 1

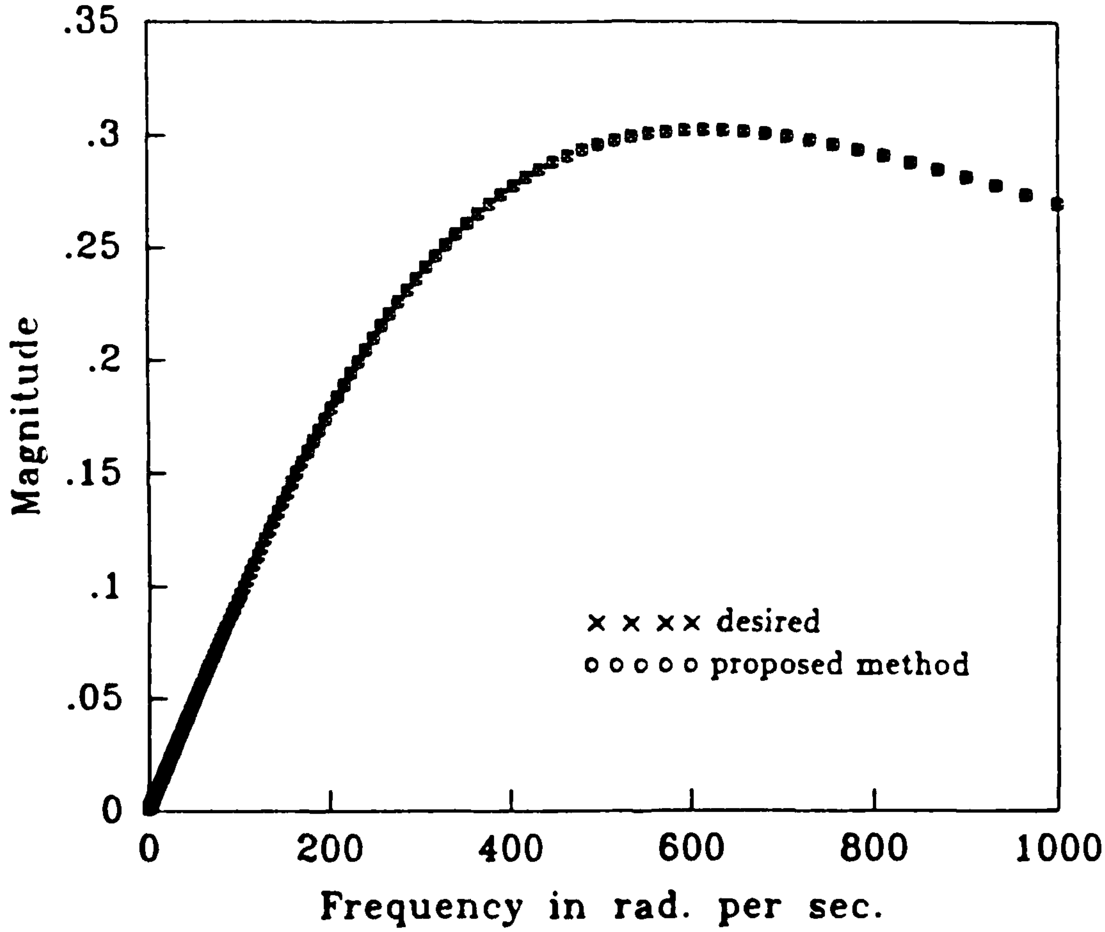

Figure 4.13: Linear magnitude response of the (2,1) element of the closed loop transfer function matrix with first order controller for Example 1

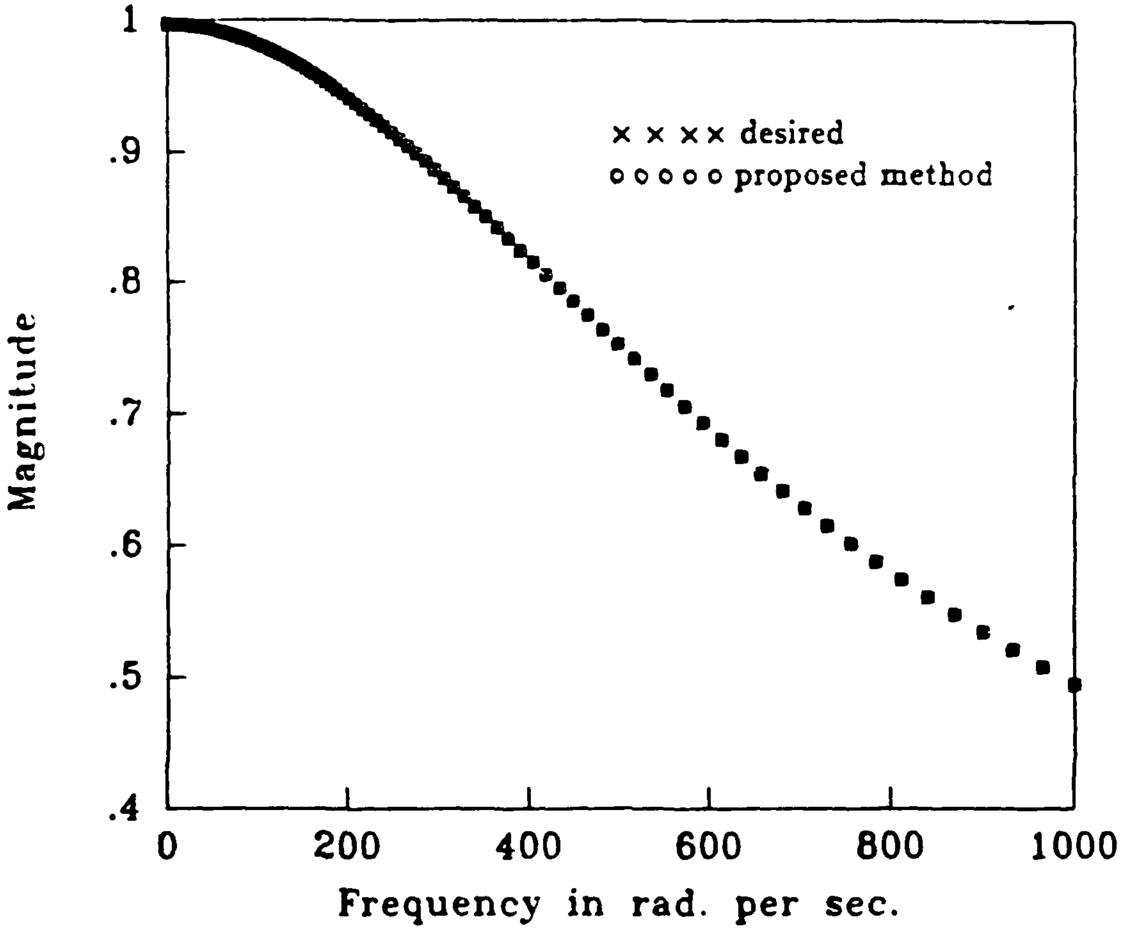

Figure 4.14: Linear magnitude response of the (2,2) element of the closed loop transfer function matrix with first order controller for Example 1

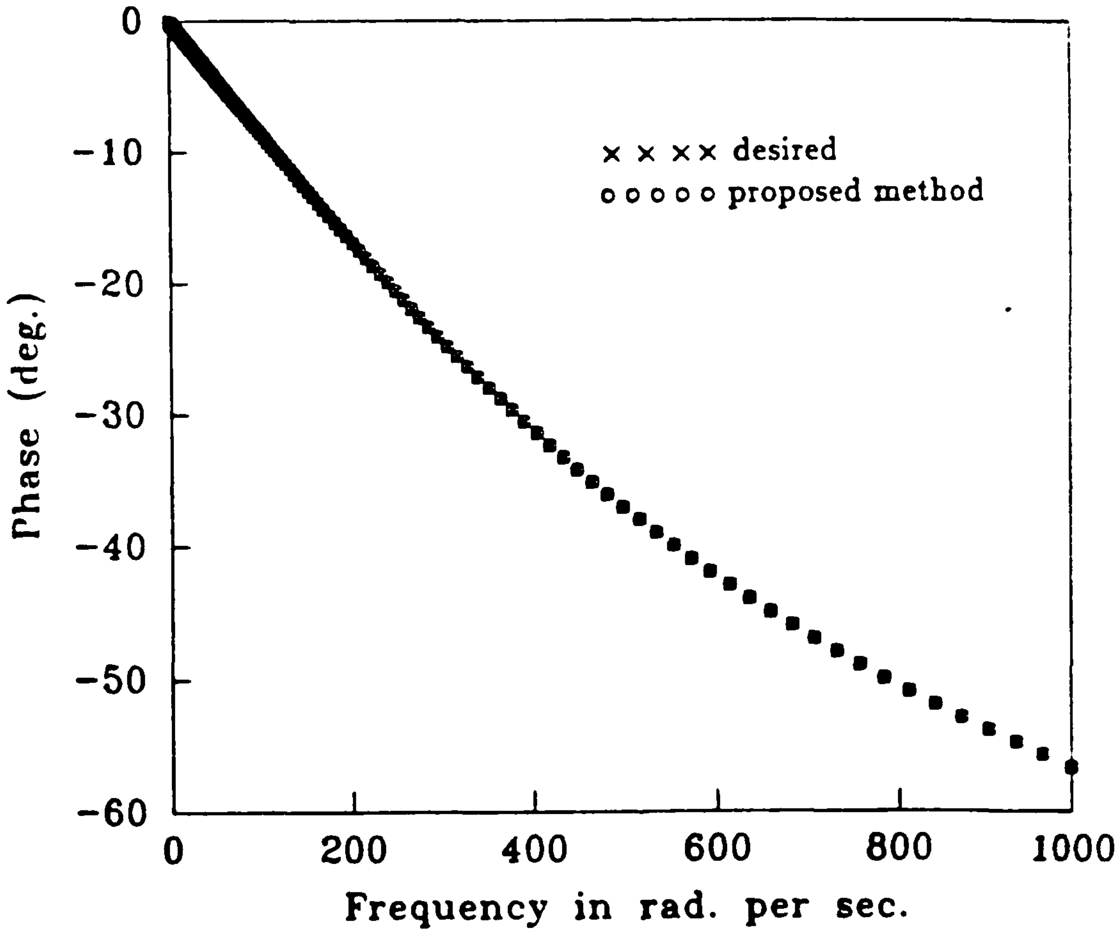

Figure 4.15: Phase response of the (1,1) element of the closed loop transfer function matrix with first order controller for Example 1

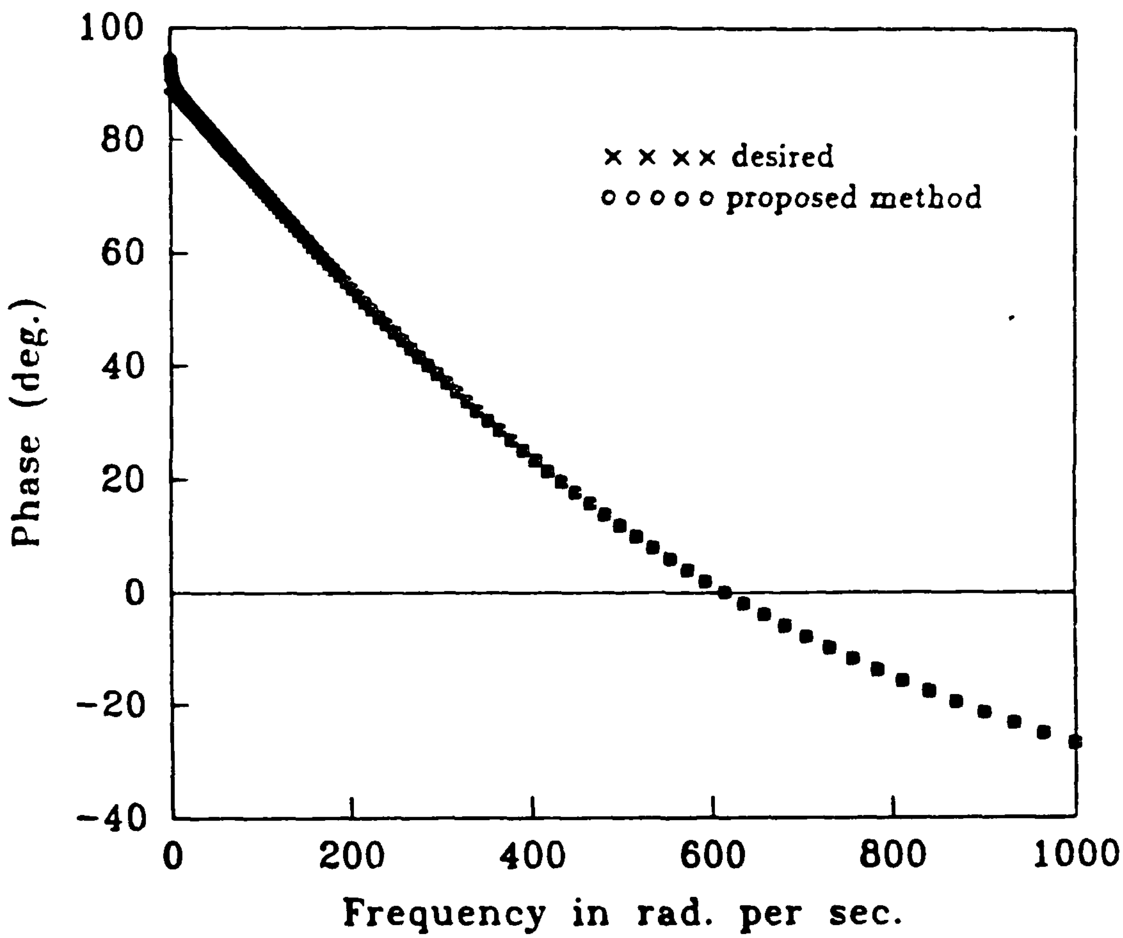

Figure 4.16: **Phase response of the (1,2) element of the closed loop transfer function matrix with first order controller for Example 1**

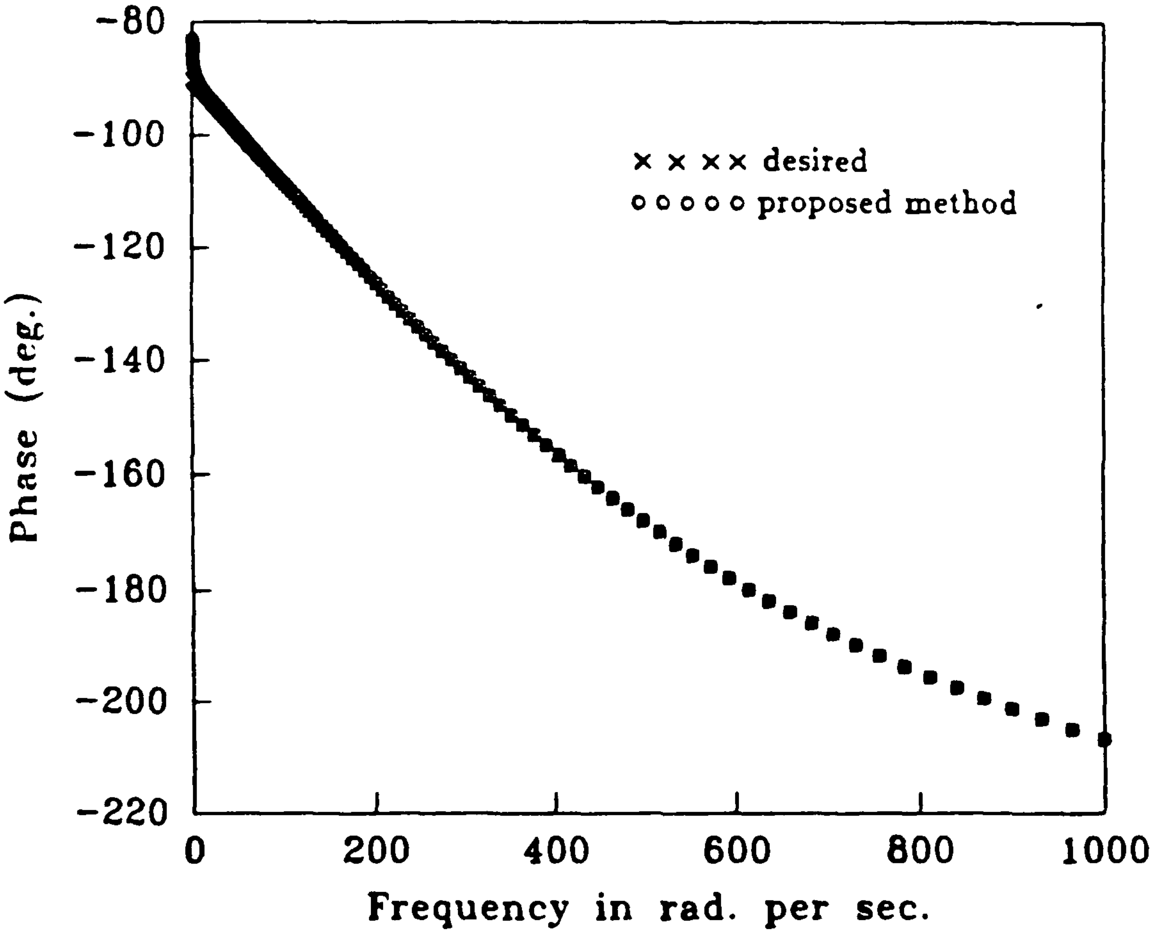

Figure 4.17: Phase response of the (2,1) element of the closed loop transfer function matrix with first order controller for Example 1

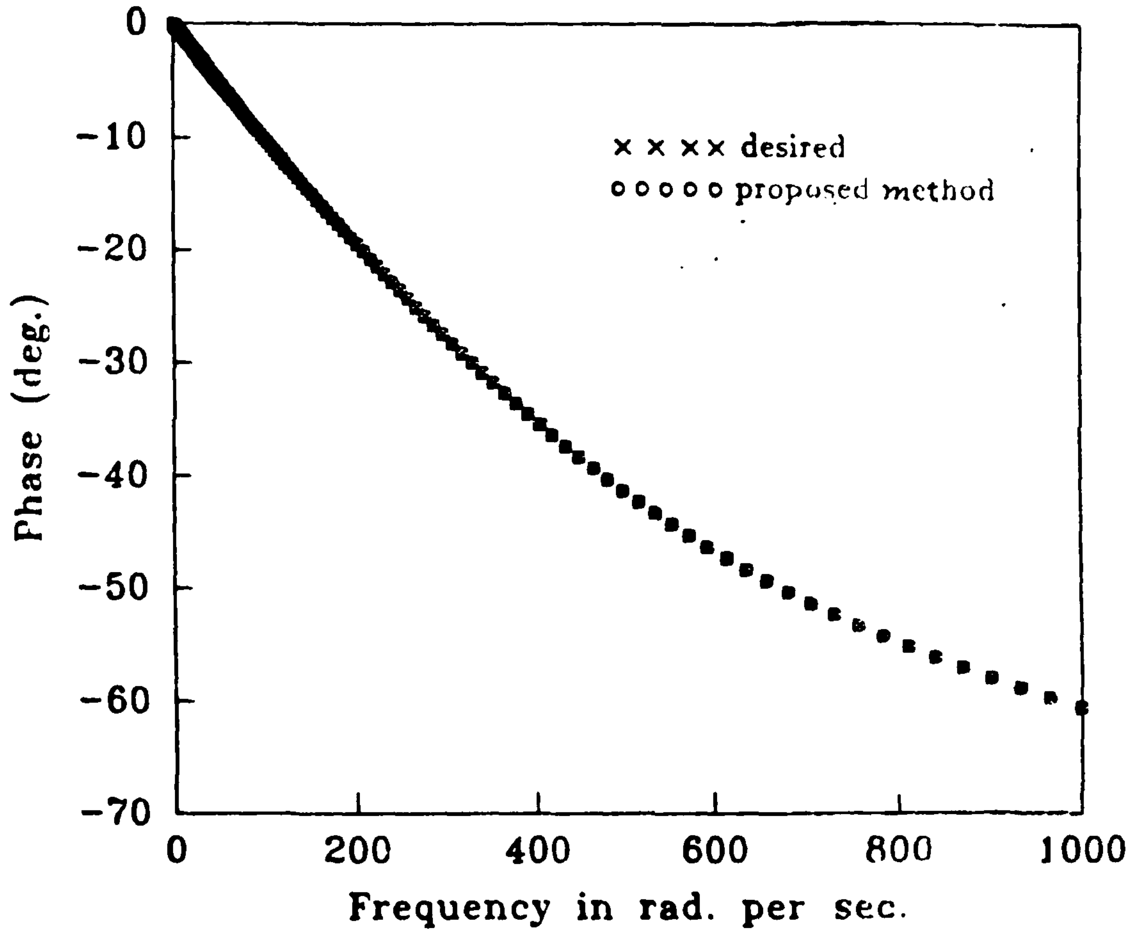

Figure 4.18: Phase response of the (2,2) element of the closed loop transfer function matrix with first order controller for Example 1

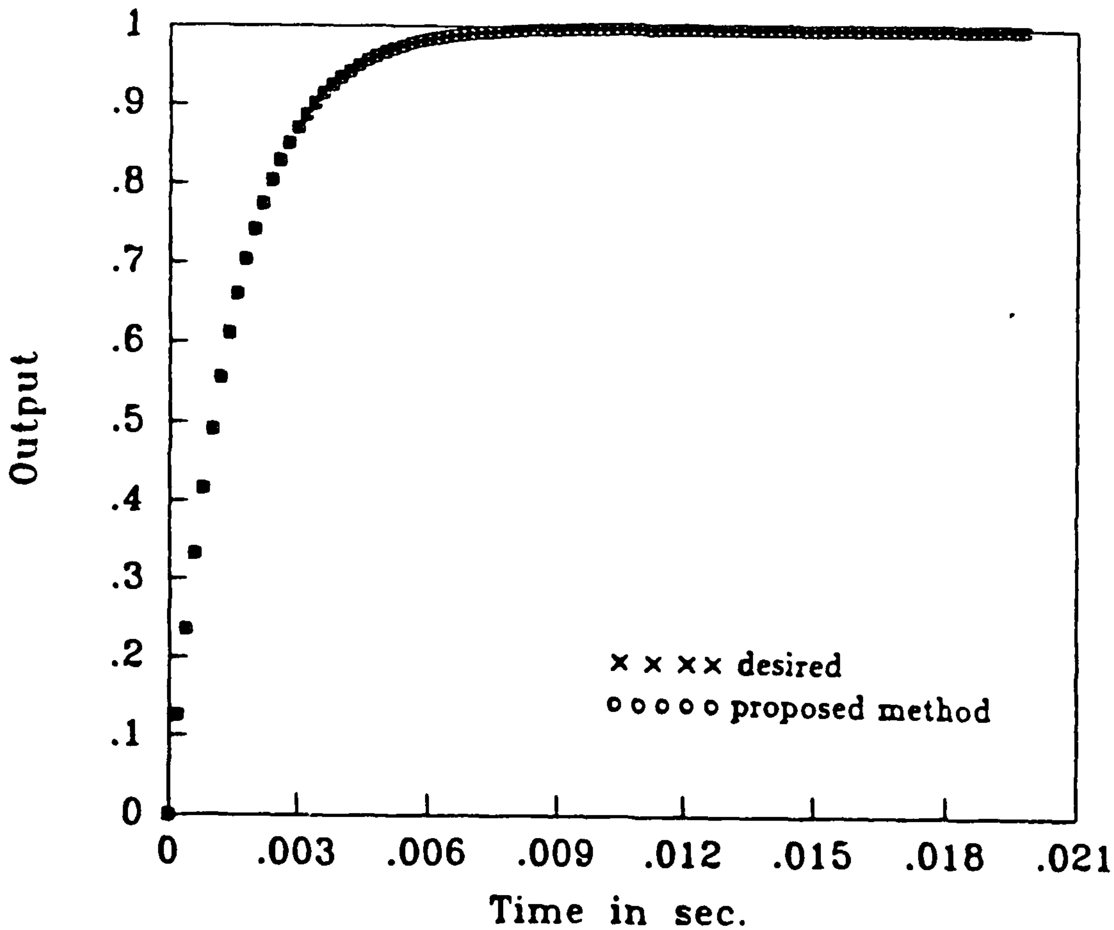

Figure 4.19: Time response of the first output of the closed loop system with first order controller for Example 1

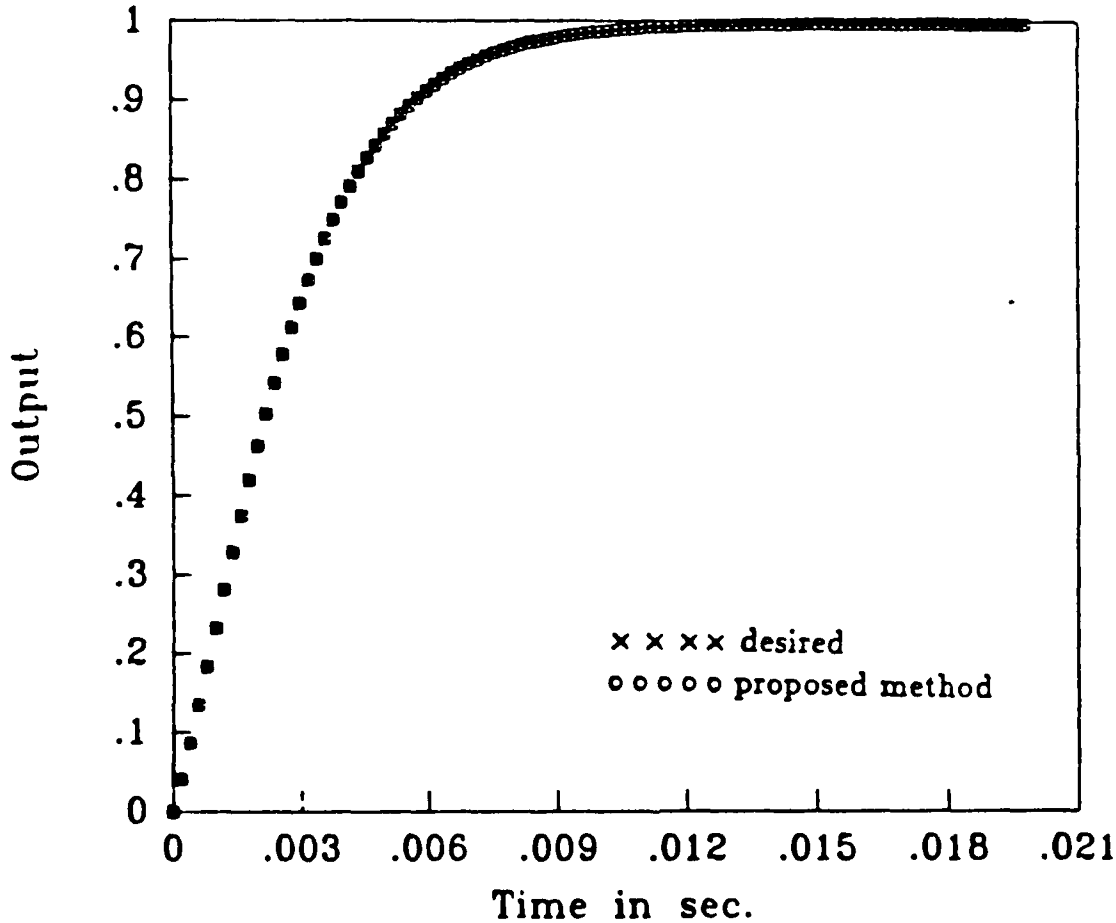

Figure 4.20: Time response of the second output of the closed loop system with first order controller for Example 1

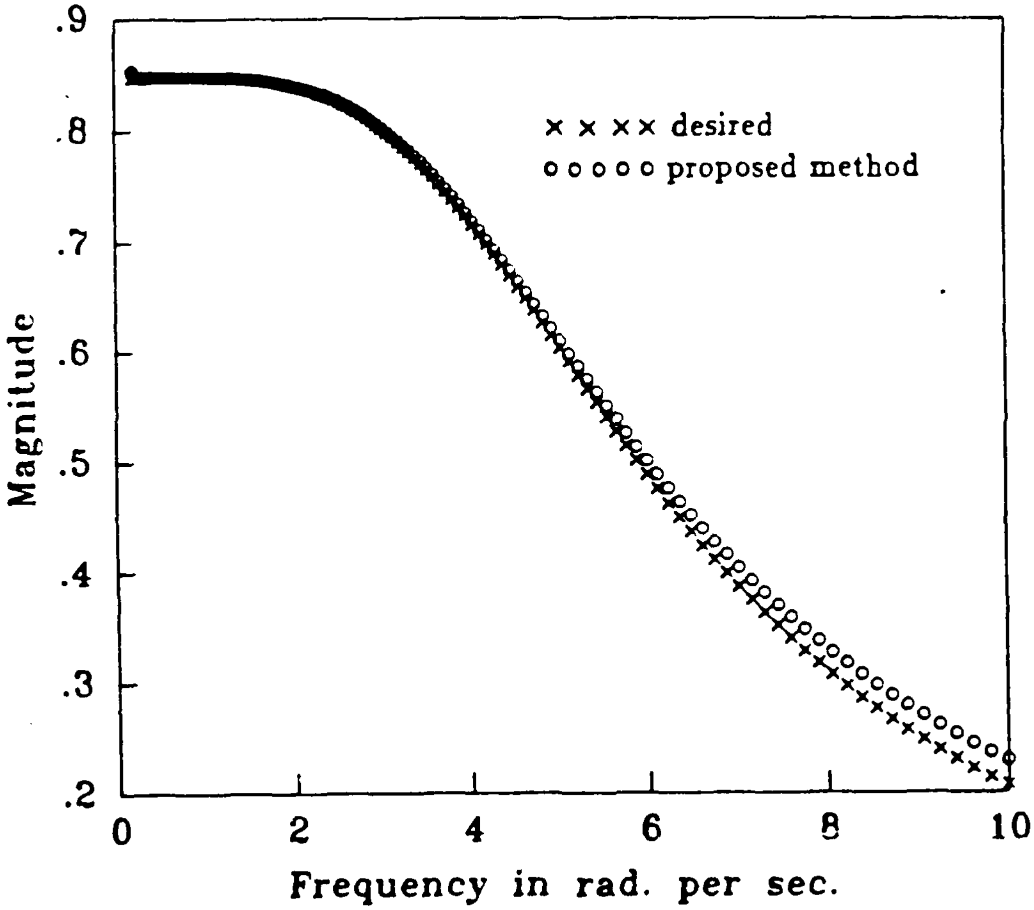

Figure 4.25: Linear magnitude response of the (1,1) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV

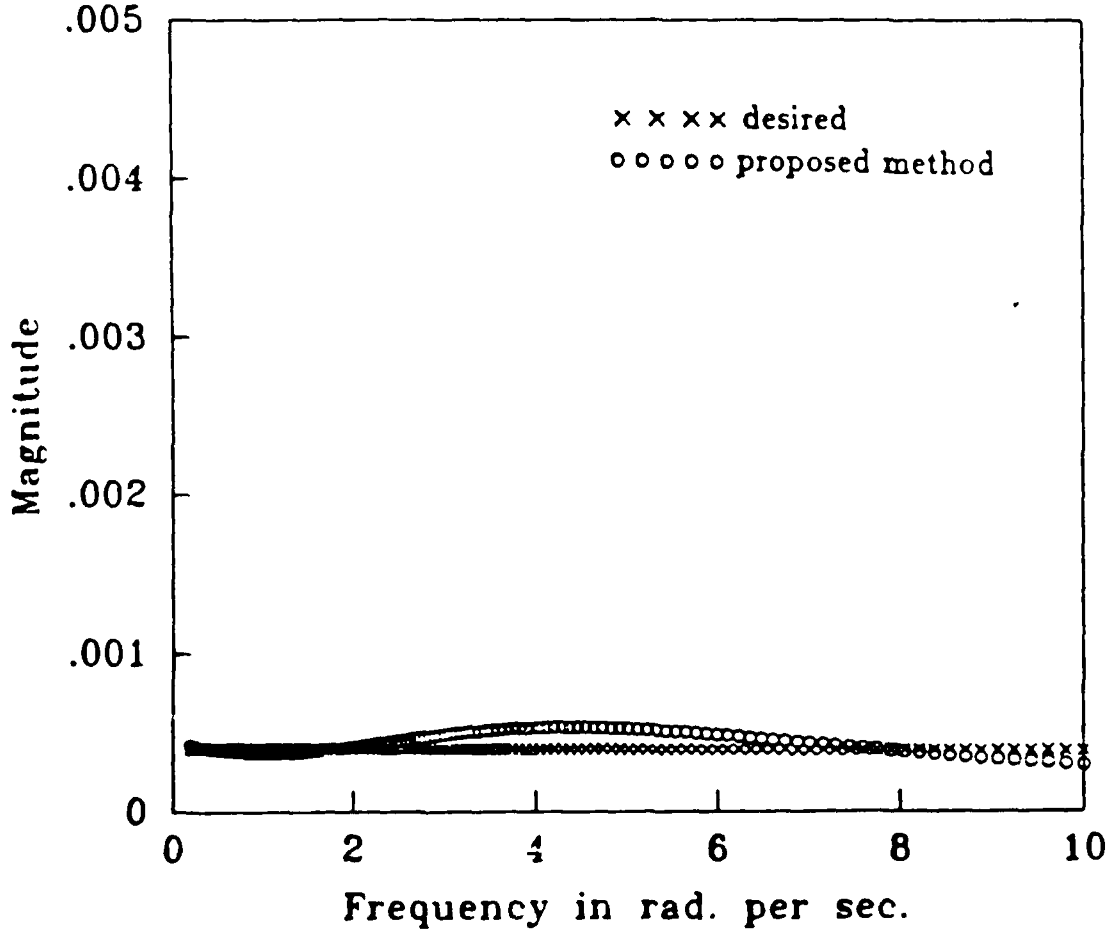

Figure 4.26: Linear magnitude response of the (1,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV

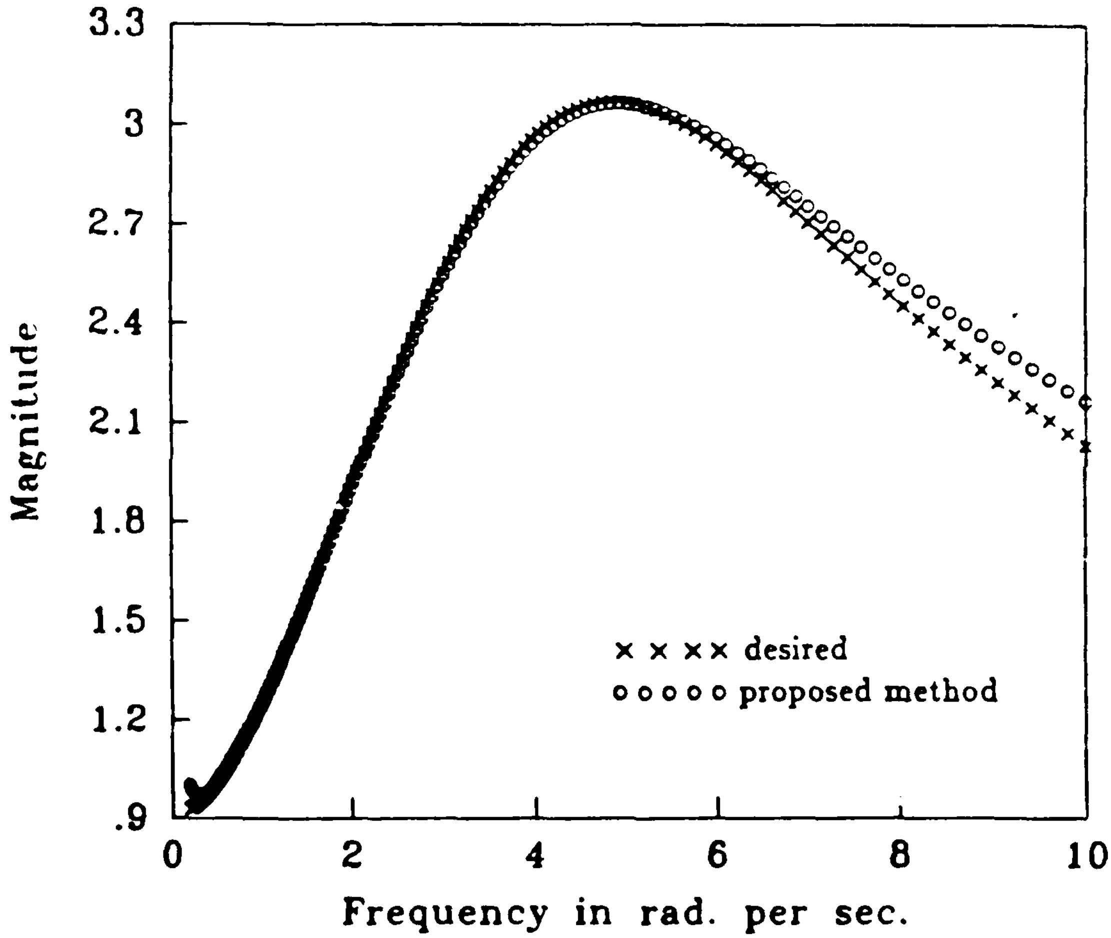

Figure 4.27: Linear magnitude response of the (2,1) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV

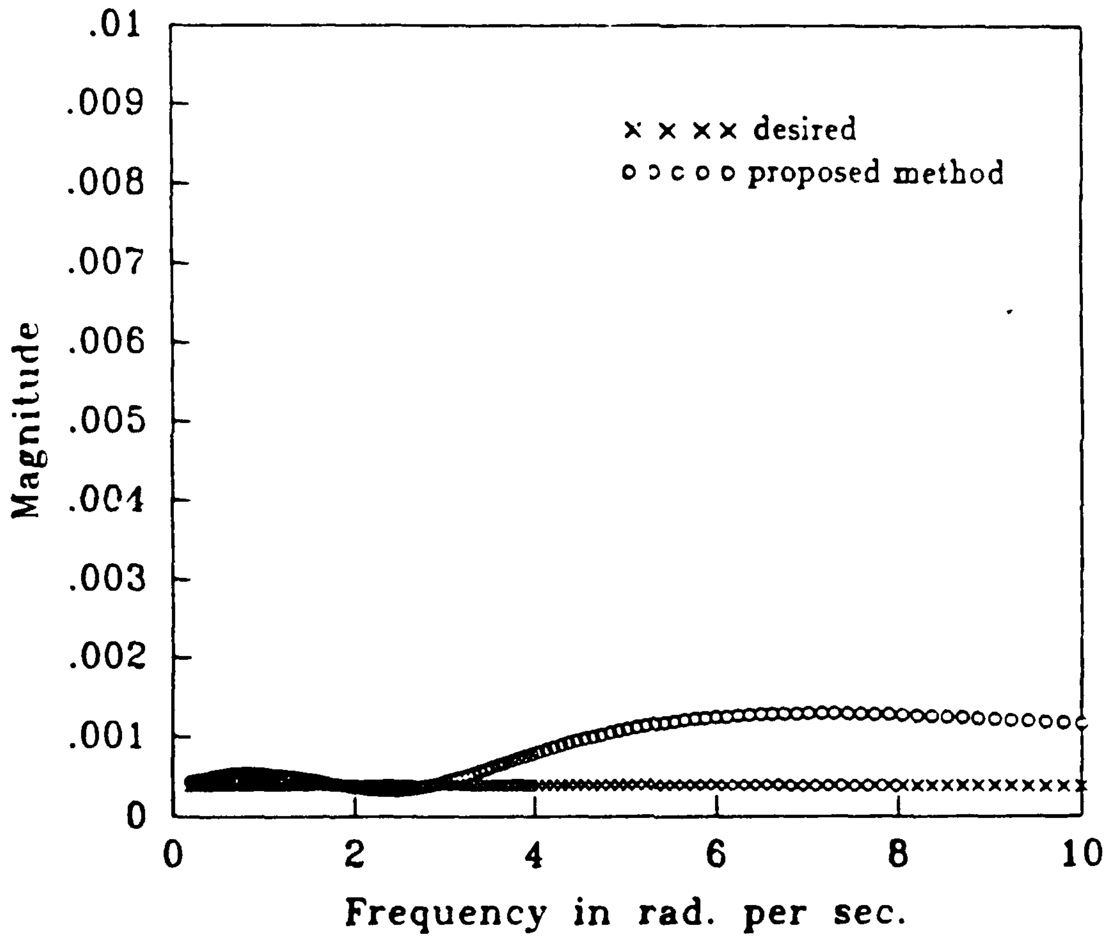

Figure 4.28: Linear magnitude response of the (2,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV

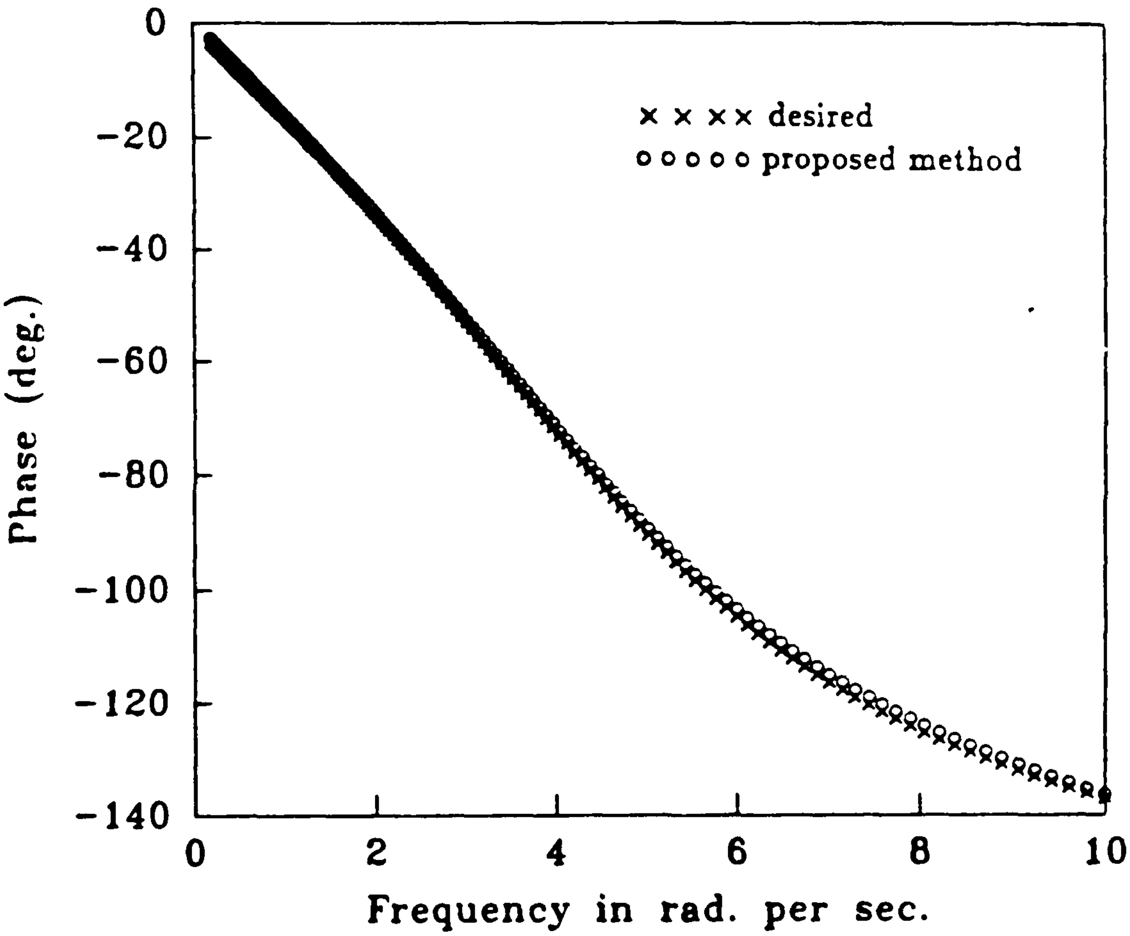

Figure 4.29: Phase response of the (1,1) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV

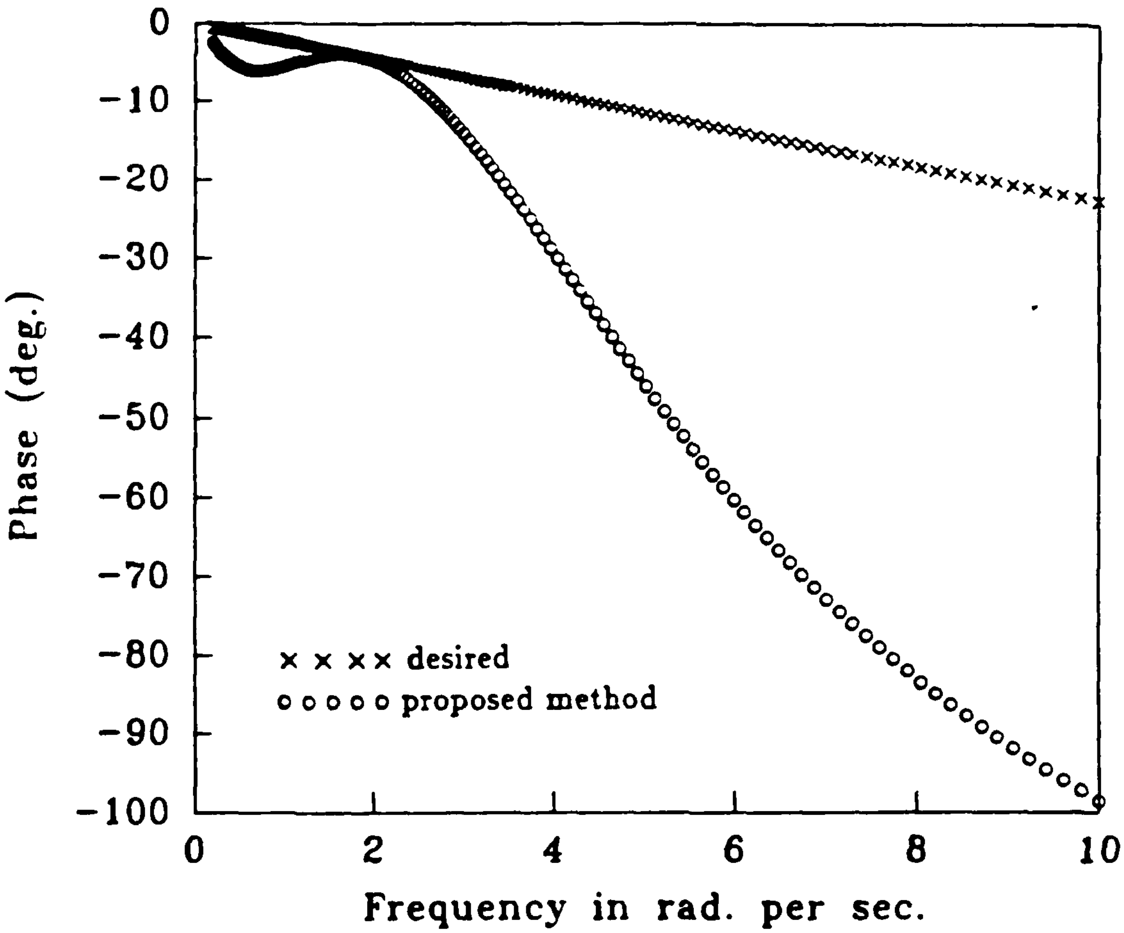

Figure 4.30: Phase response of the (1,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV

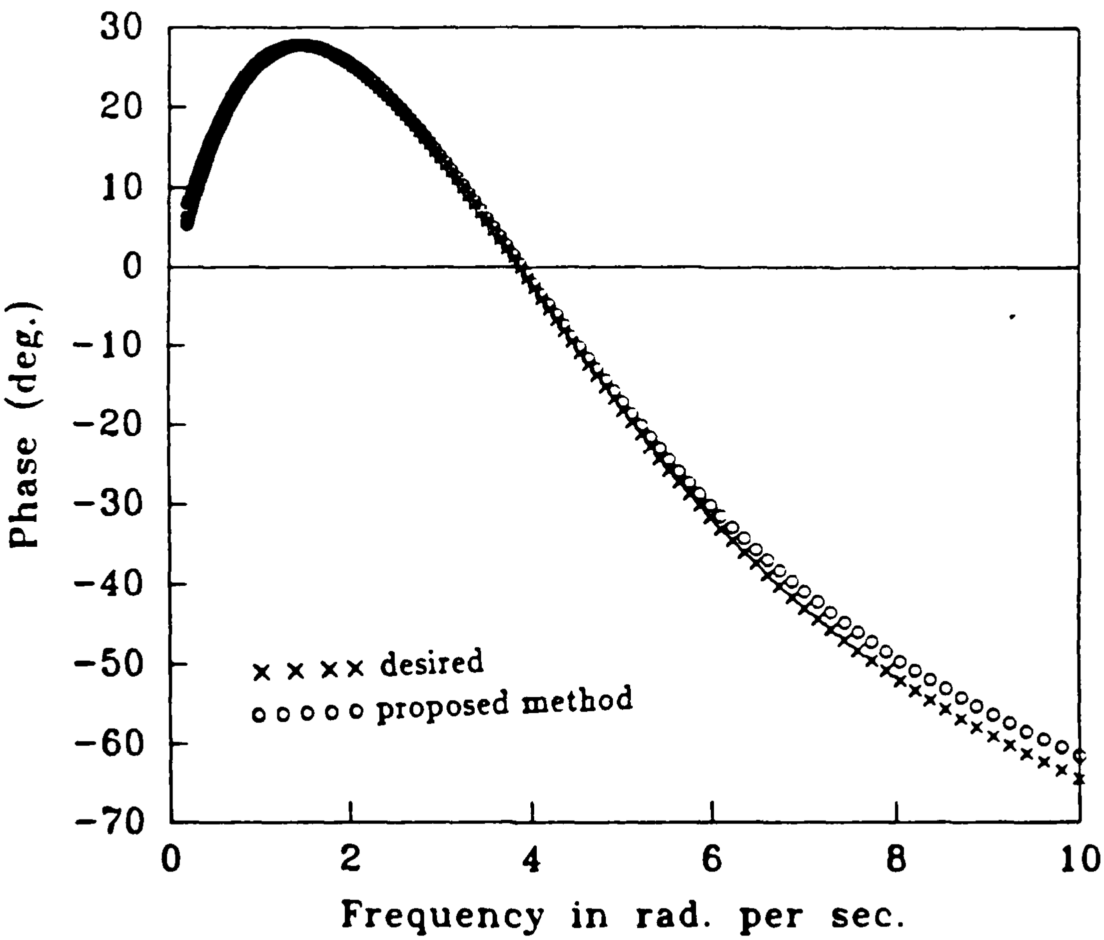

Figure 4.31: Phase response of the (2,1) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV

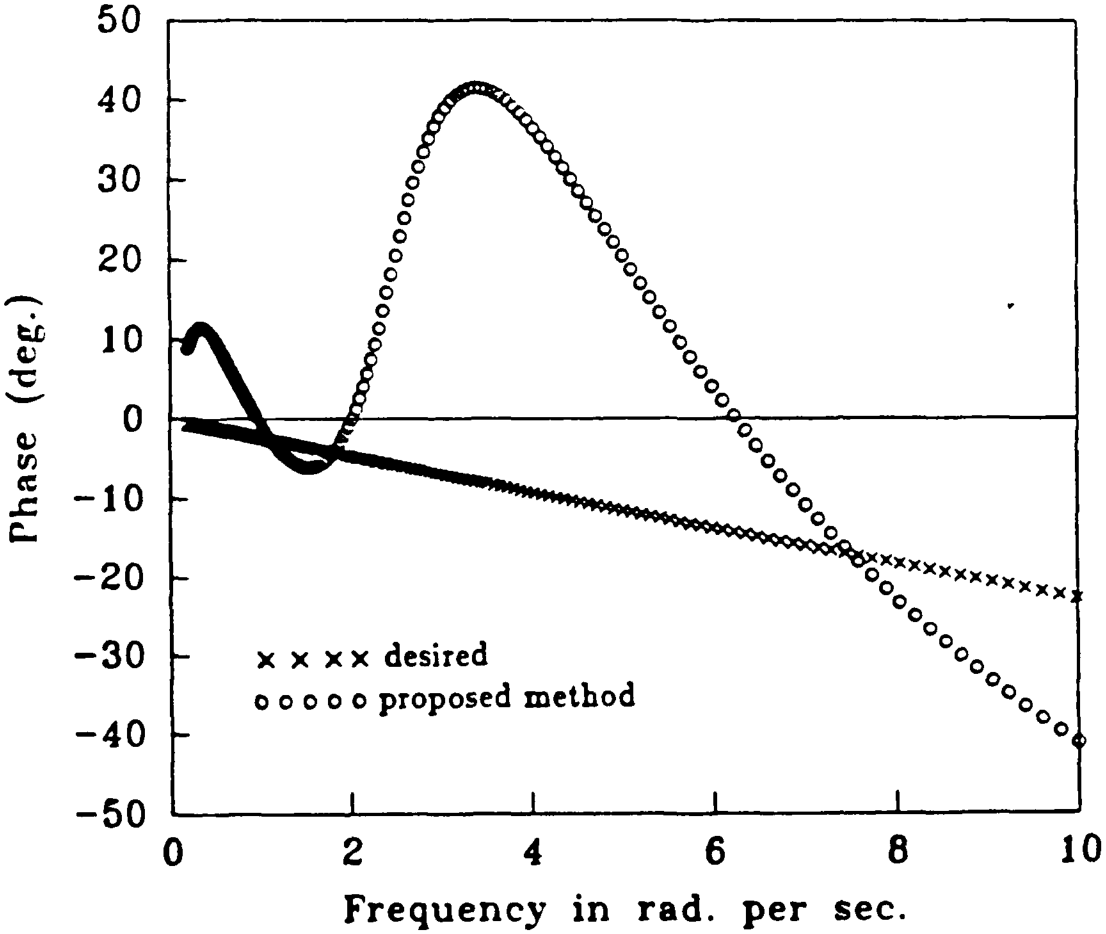

Figure 4.32: Phase response of the (2,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV

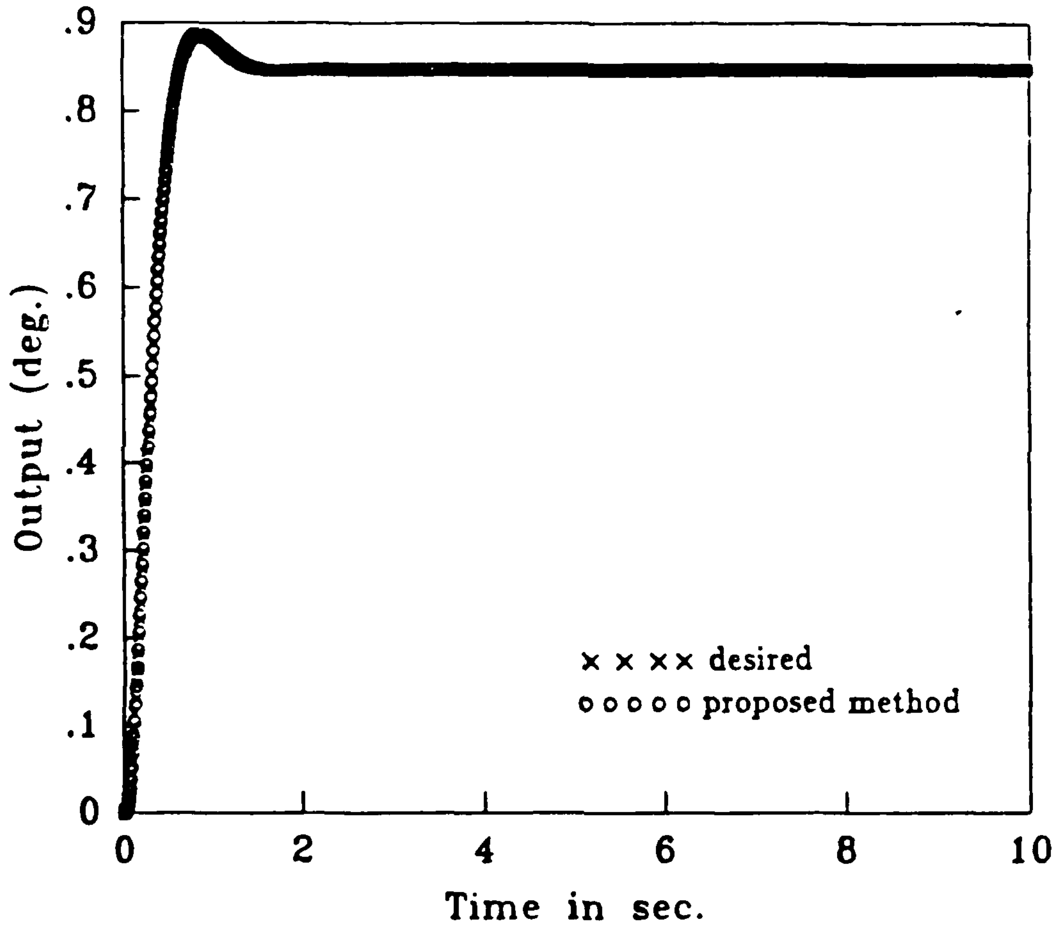

Figure 4.33: Time response of the angle of attack of the compensated system and the 'desired' system for YF-16 CCV

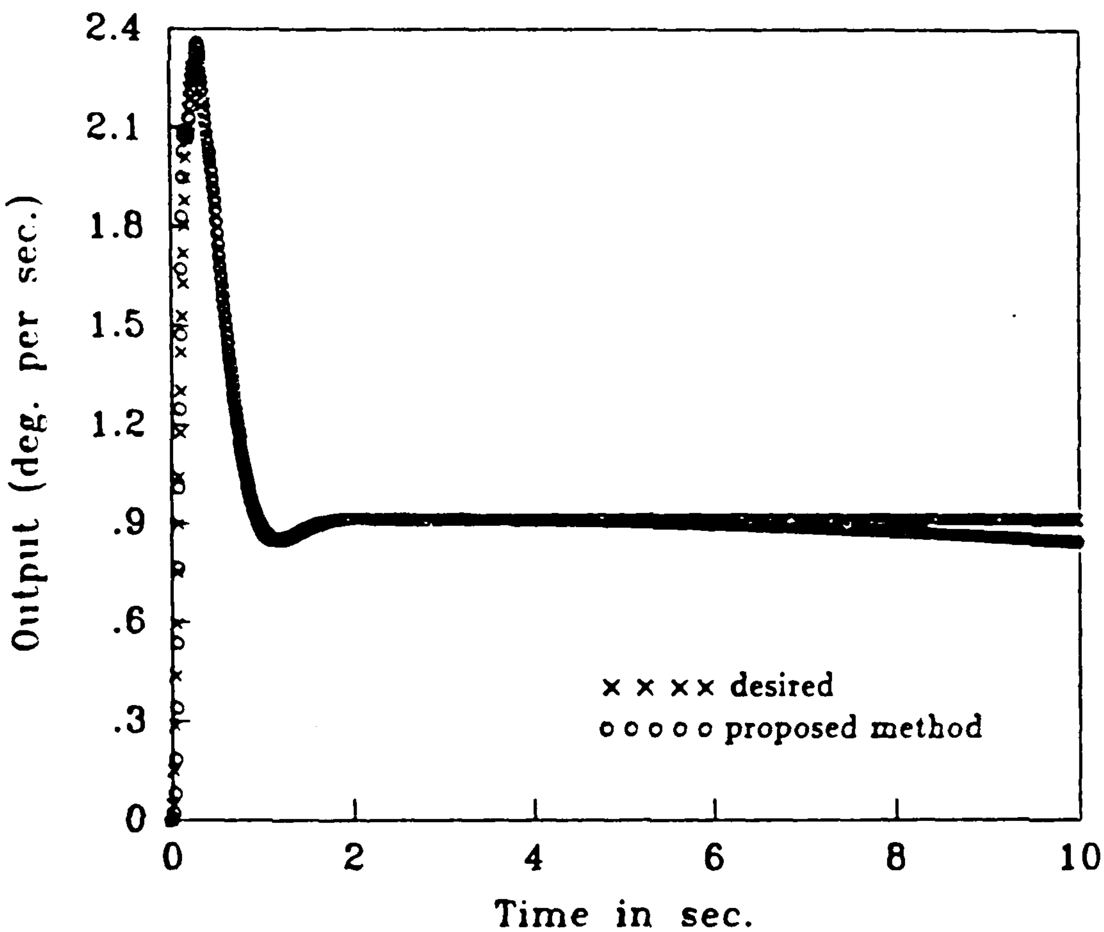

Figure 4.34: Time response of the pitch rate of the compensated system and the 'desired' system for YF-16 CCV

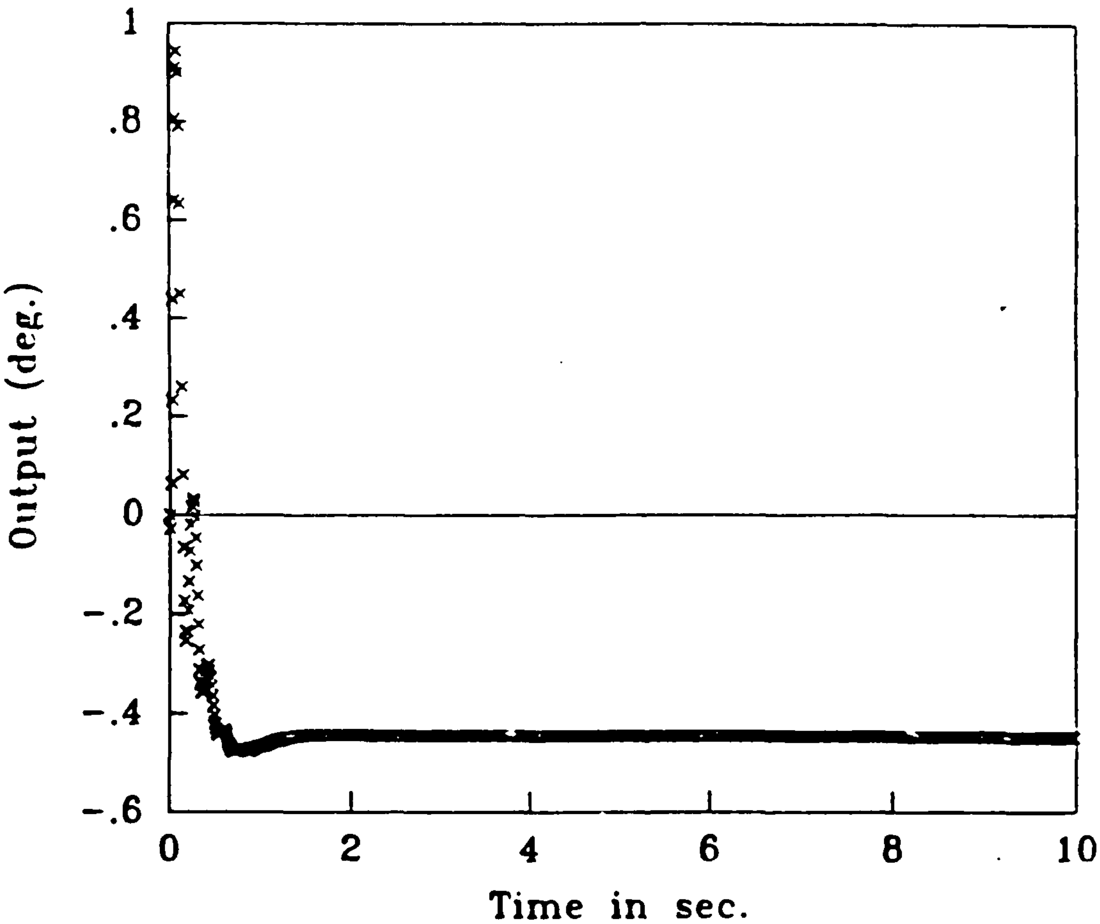

Figure 4.35: Time response of the elevator deflection of the compensated system for YF-16 CCV

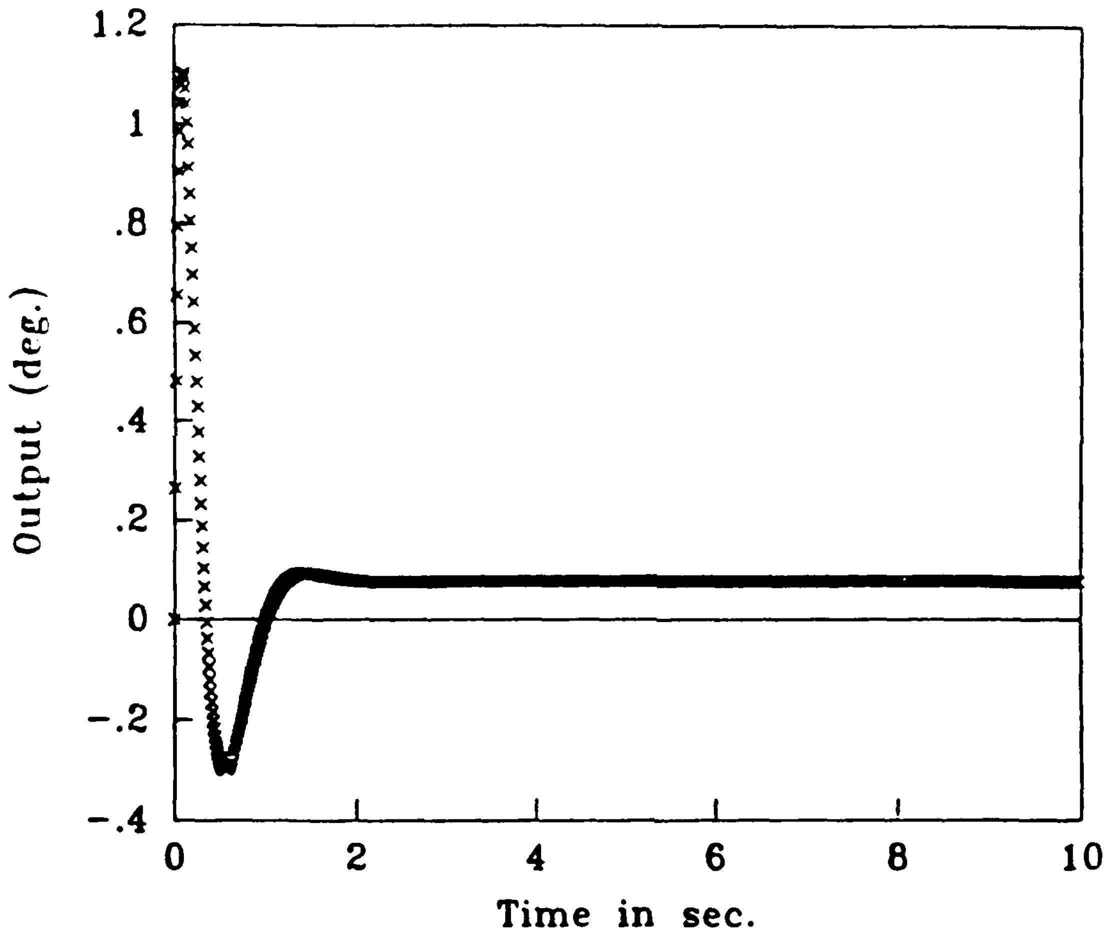

Figure 4 36: Time response of the flap deflection of the compensated system for YF-16 CCV

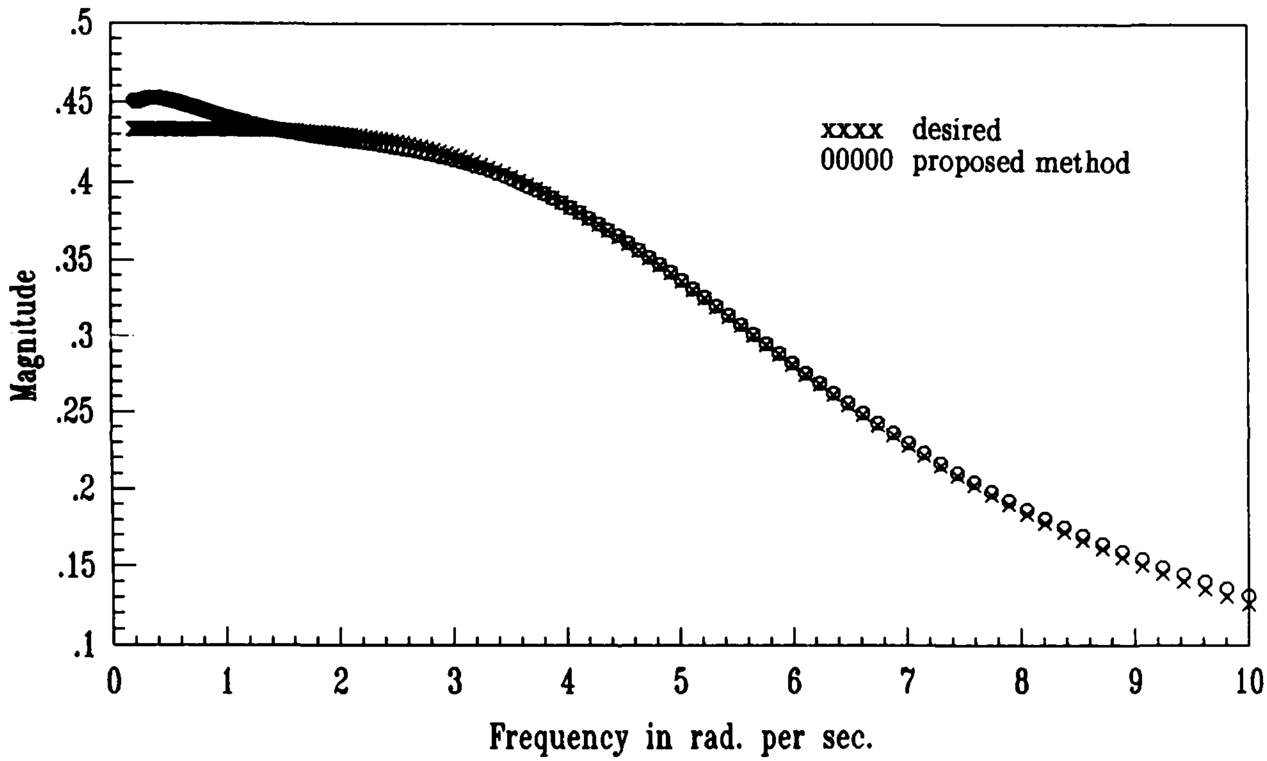

Figure 4.37: Linear magnitude response of the (1,1) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, CASE 2.

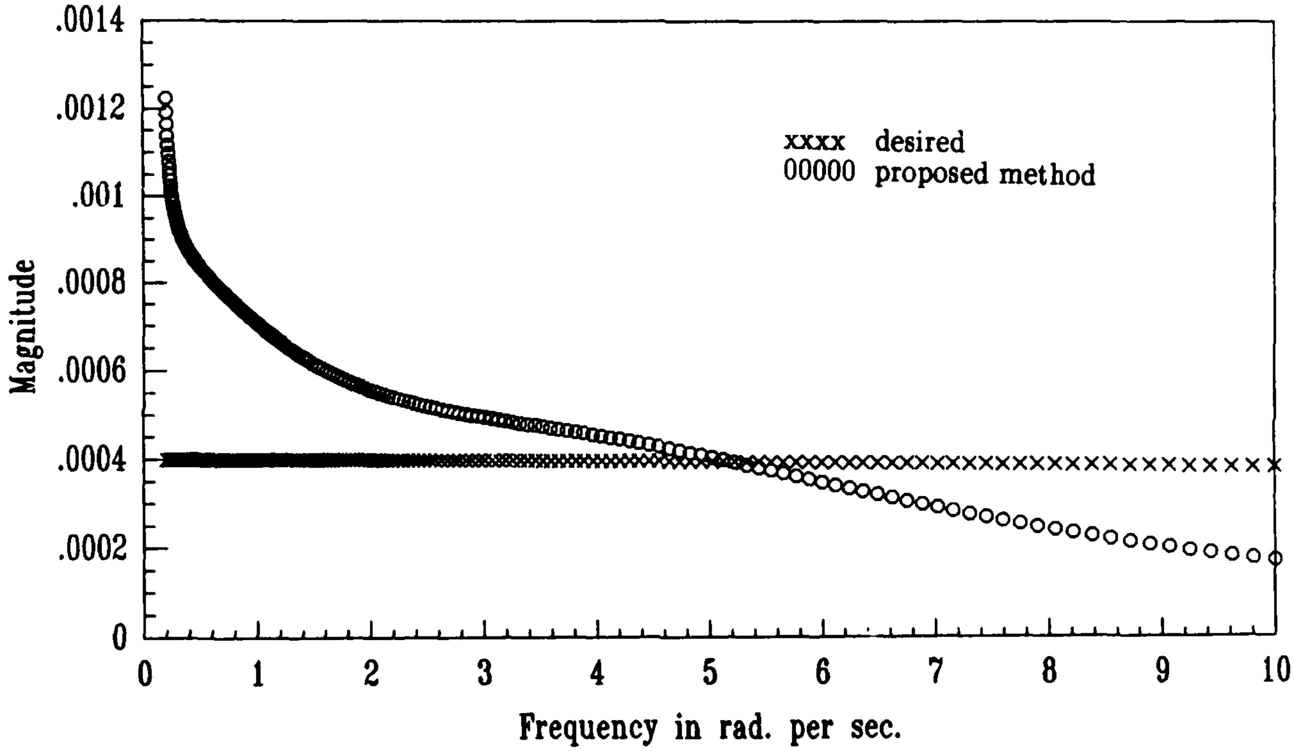

Figure 4.38 : Linear magnitude response of the (1,2) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, CASE 2.

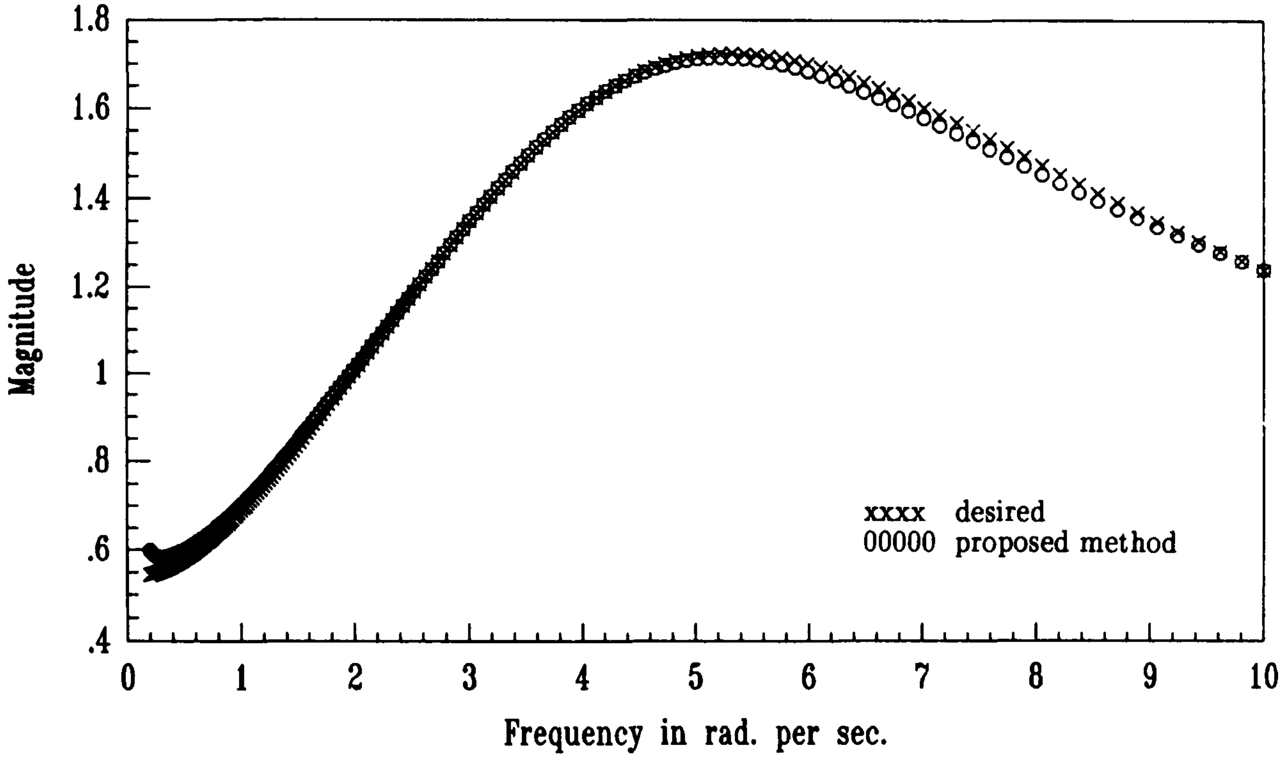

Figure 4.39: Linear magnitude response of the (2,1) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, CASE 2.

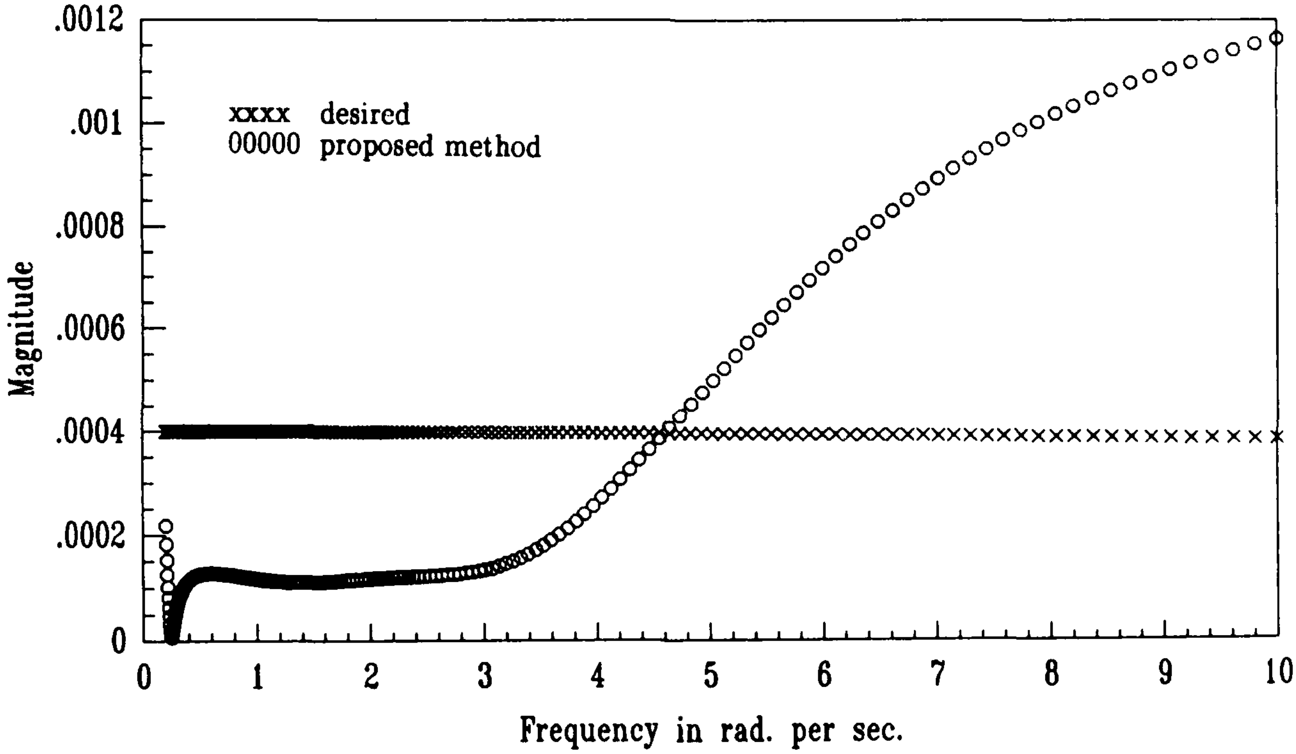

Figure 4.40: Linear magnitude response of the (2,2) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, CASE 2.

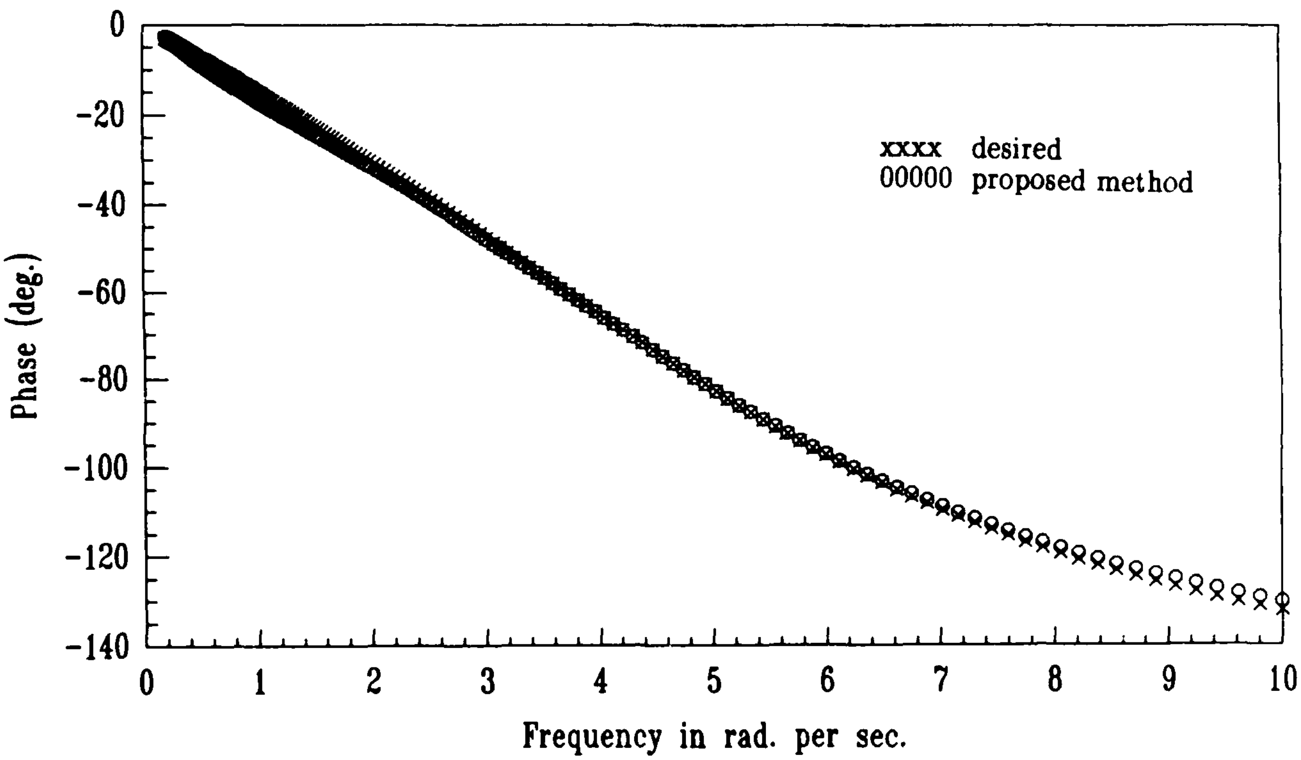

Figure 4.41: Phase response of the (1,1) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, CASE 2.

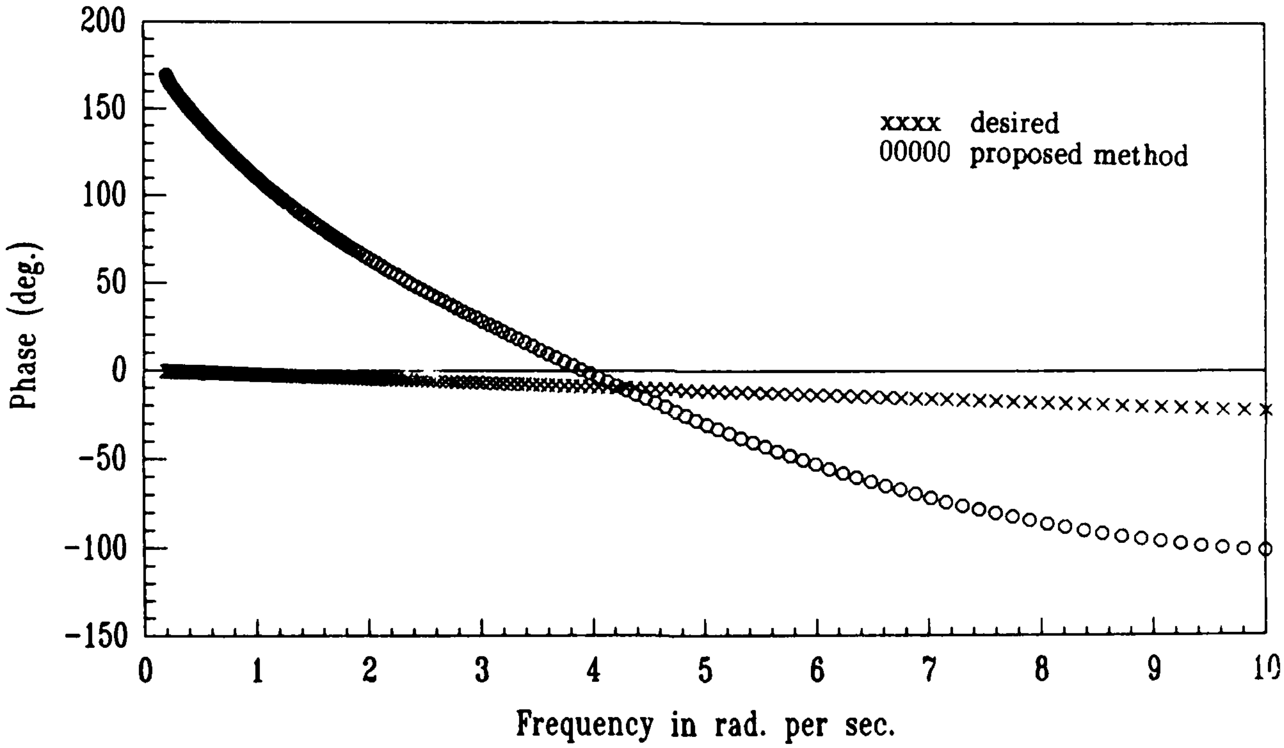

Figure 4.42: Phase response of the (1,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, CASE 2.

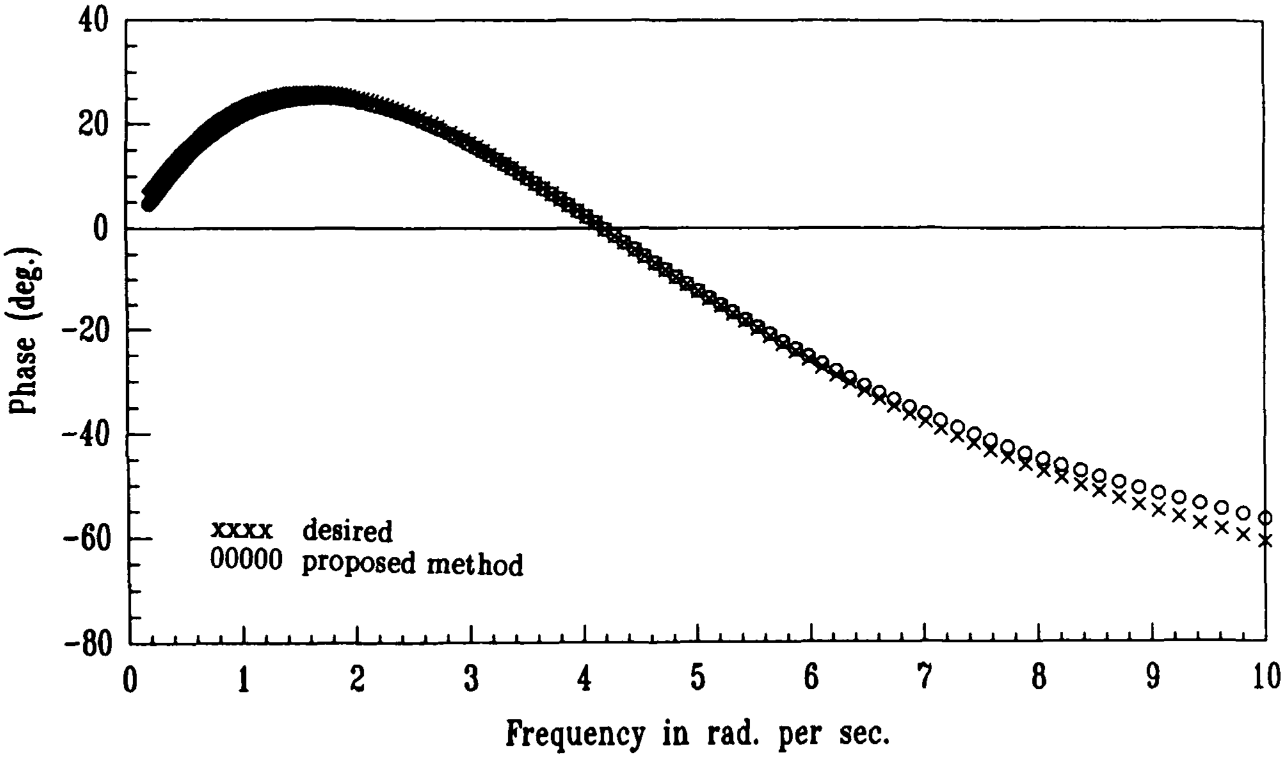

Figure 4.43: Phase response of the (2,1) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, CASE 2.

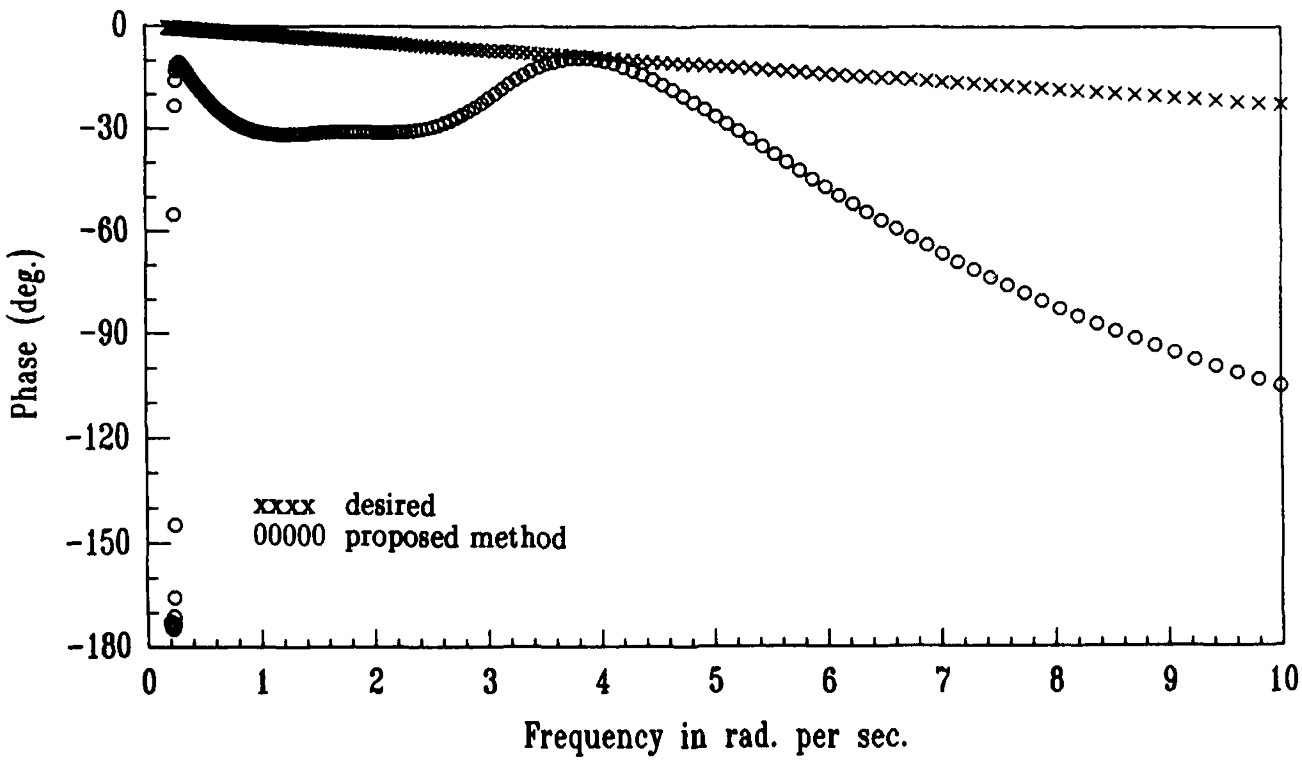

Figure 4.44: Phase response of the (2,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, CASE 2.

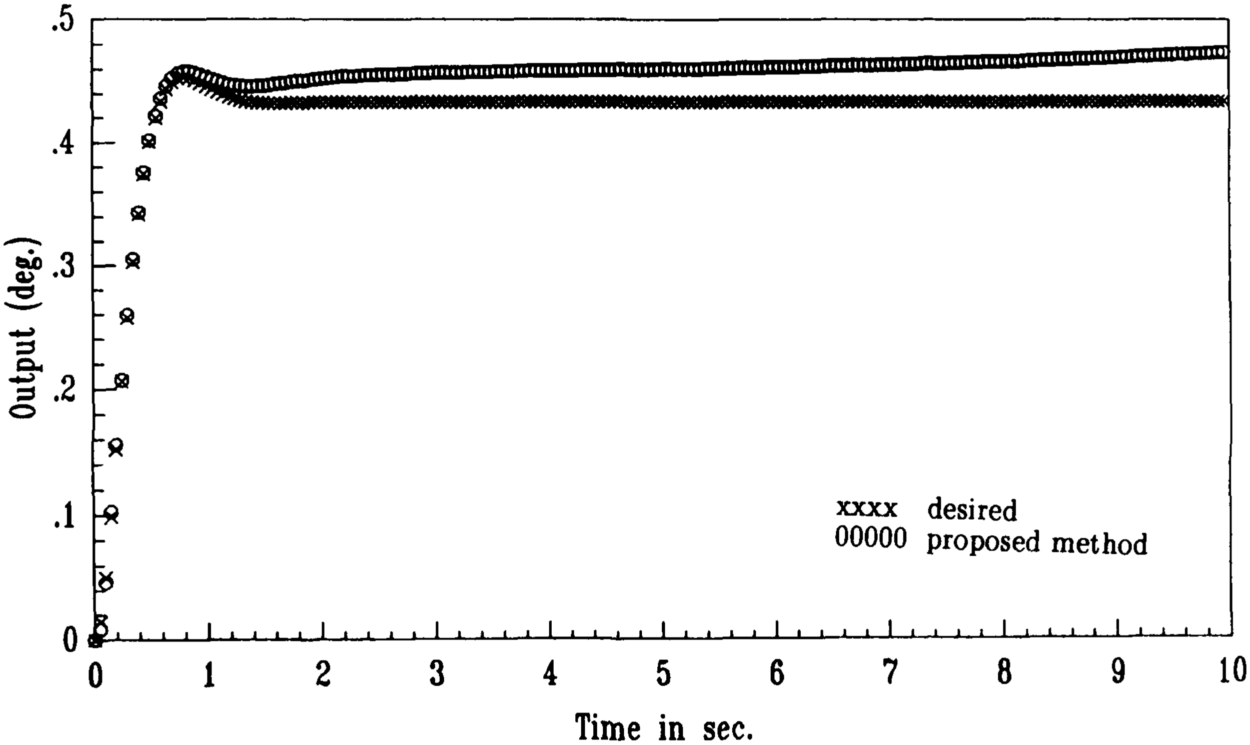

Figure 4.45 Time response of the (1,1) element of the compensated system and the 'desired' system for YF–16 CCV, CASE 2.

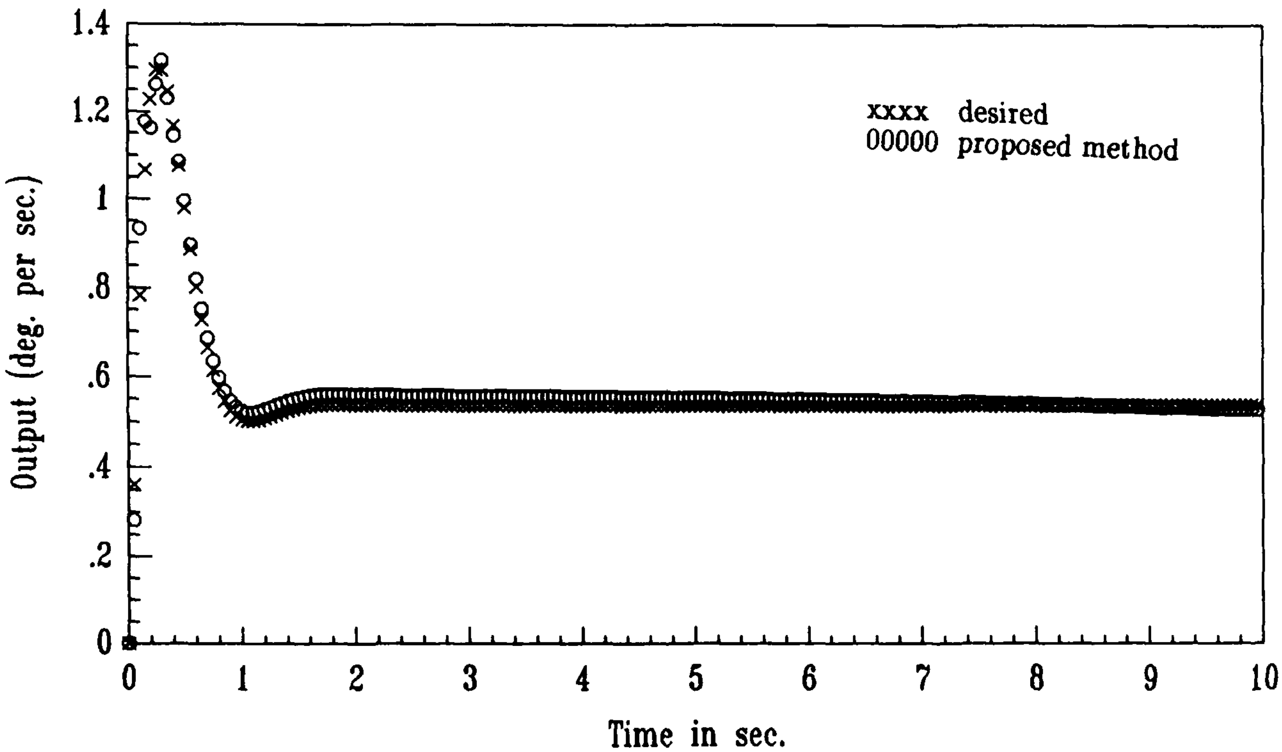

Figure 4.46: Time response of the (2,1) element of the compensated system and the 'desired' system for YF-16 CCV, CASE 2.

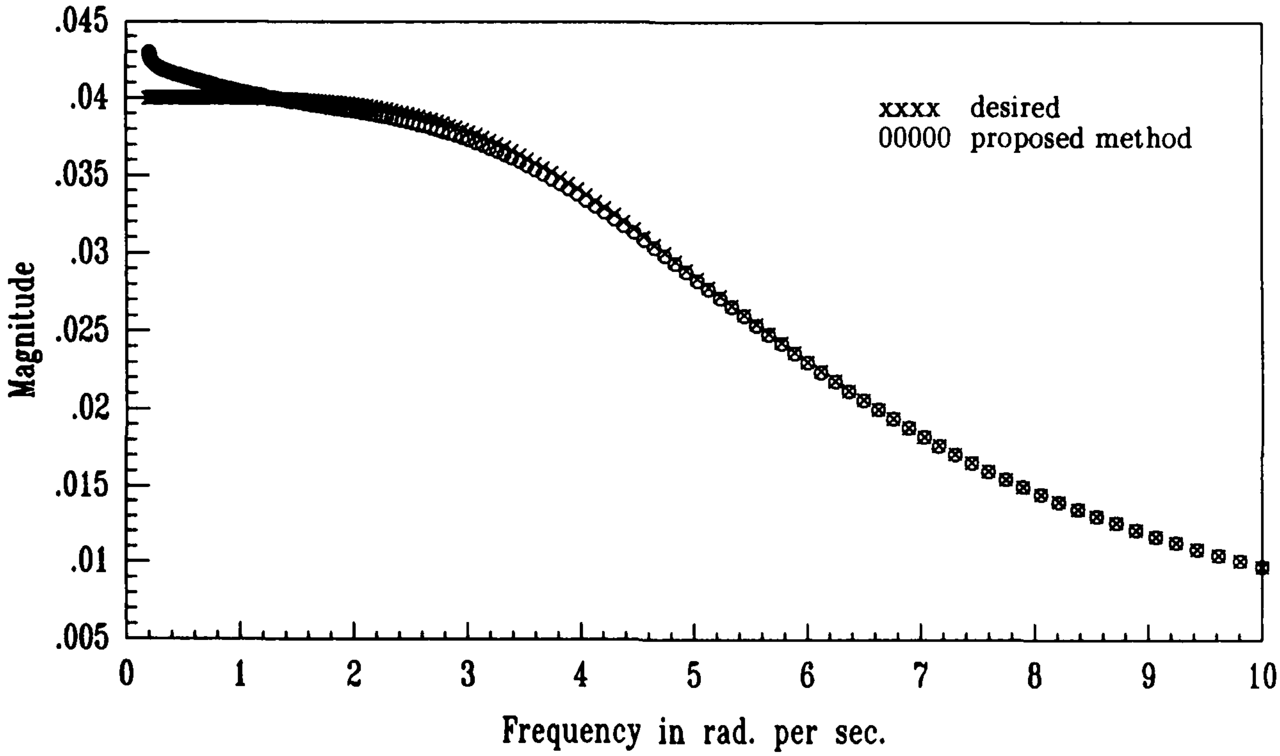

Figure 4.48: Linear magnitude response of the (1,1) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, CASE 3.

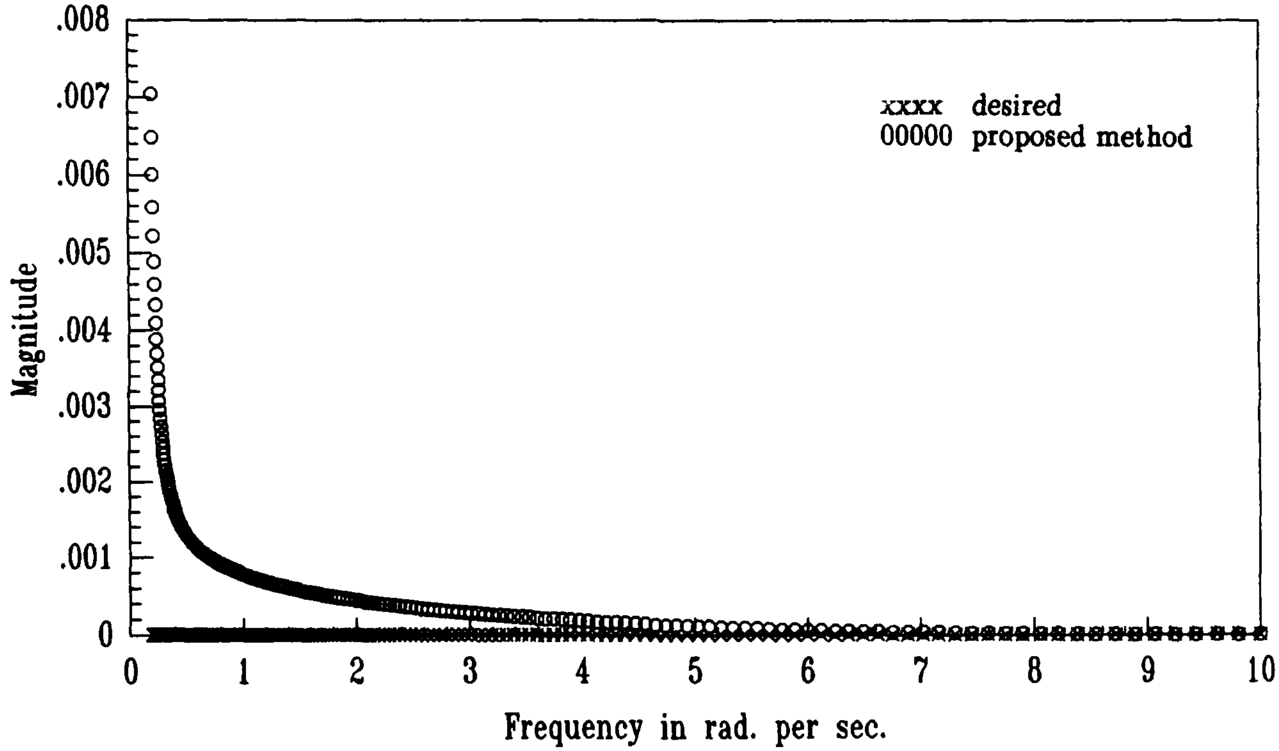

Figure 4.49: Linear magnitude response of the (1,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, CASE 3.

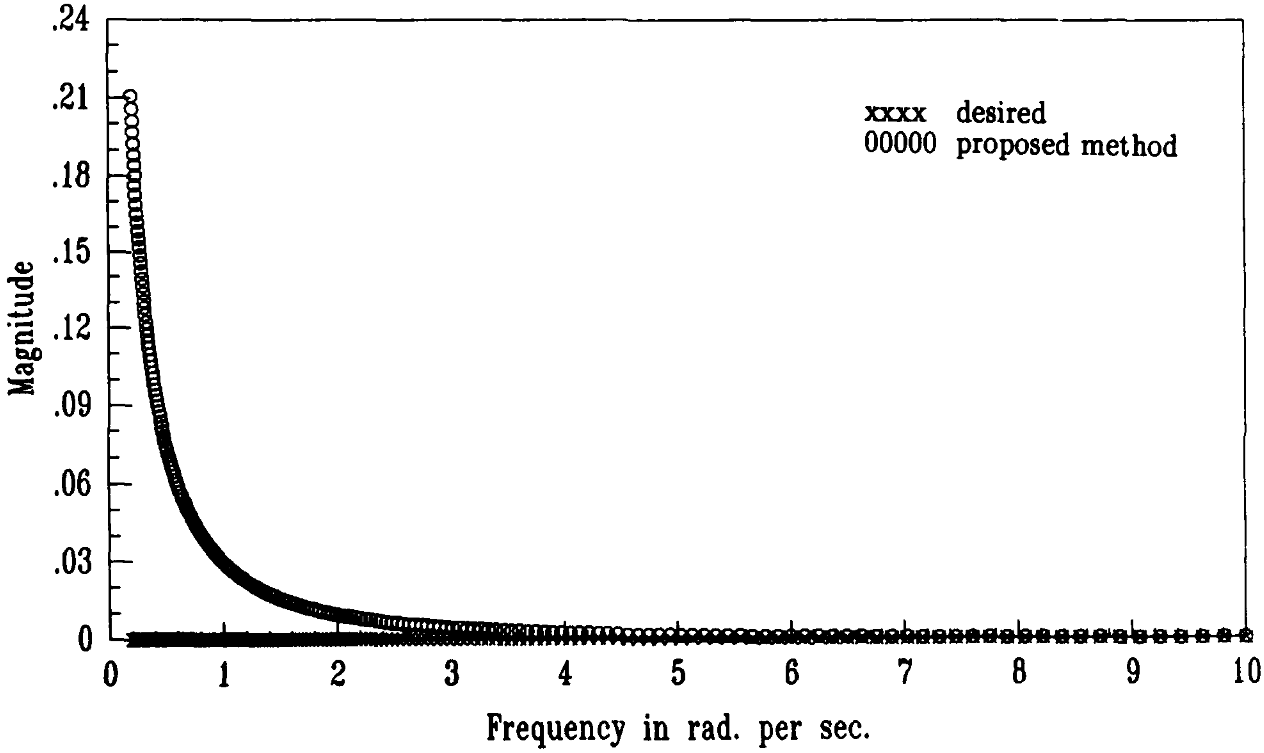

Figure 4.50: Linear magnitude response of the (2,1) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, CASE 3.

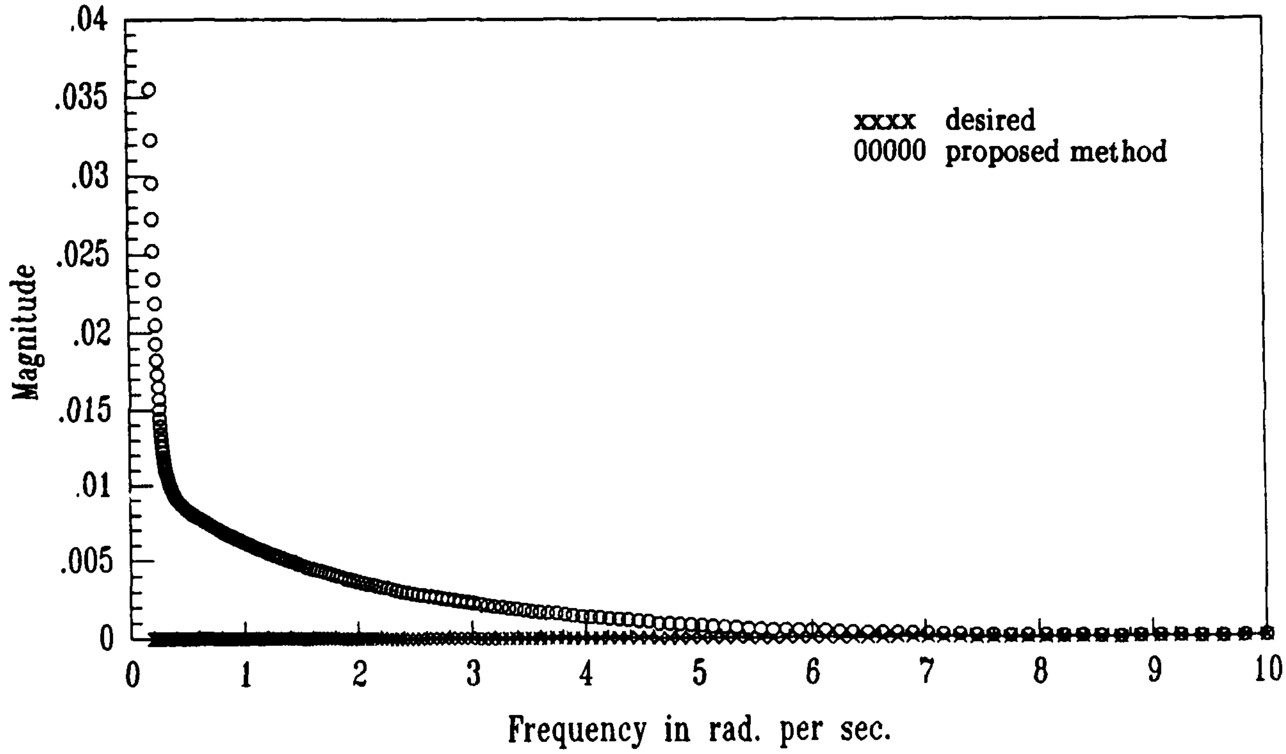

Figure 4.51: Linear magnitude response of the (2,2) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, CASE 3.

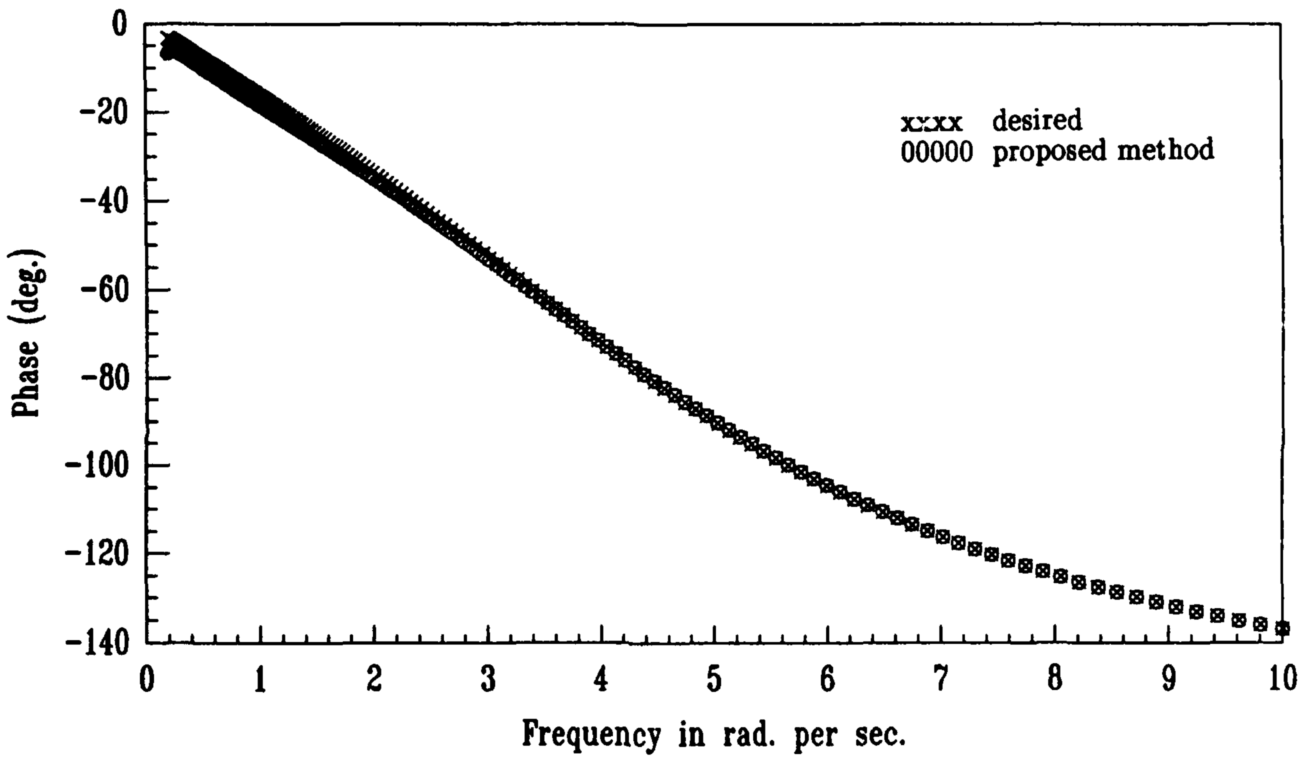

Figure 4.52: Phase response of the (1,1) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, CASE 3.

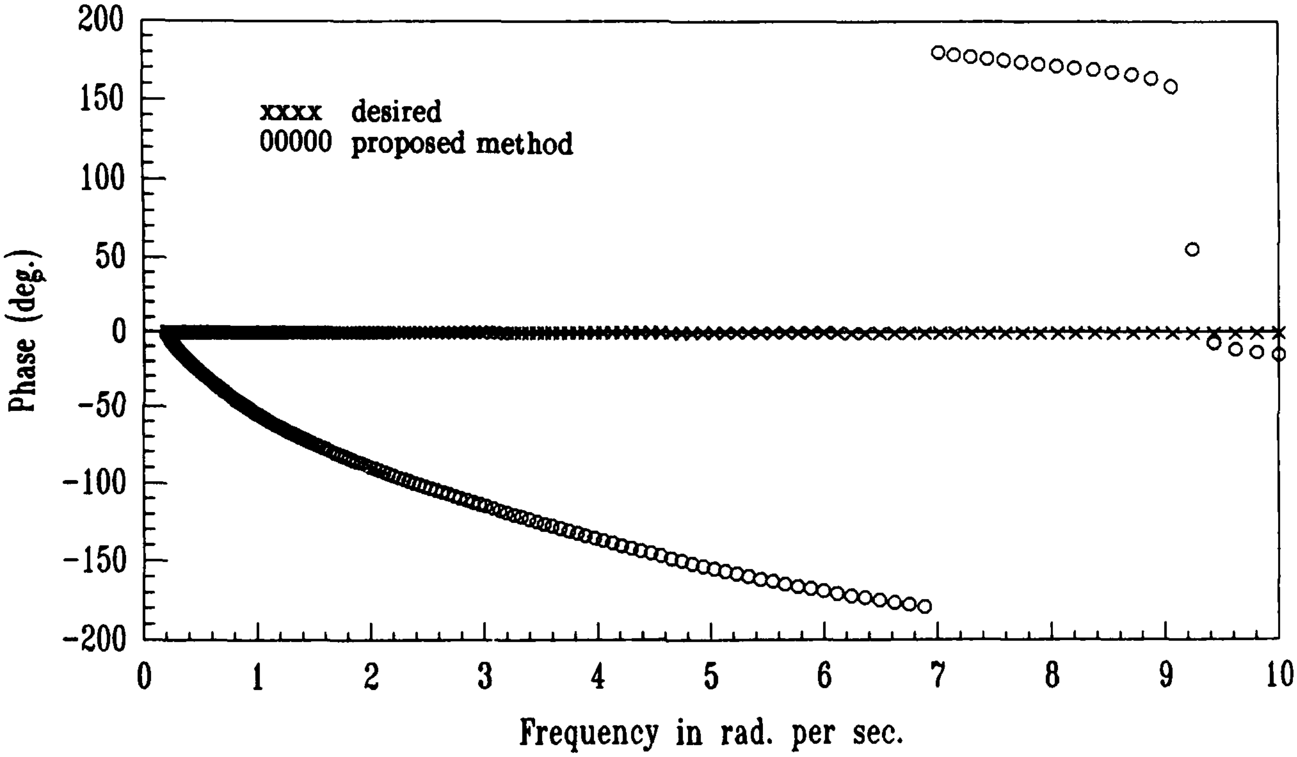

Figure 4.53: Phase response of the (1,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, CASE 3.

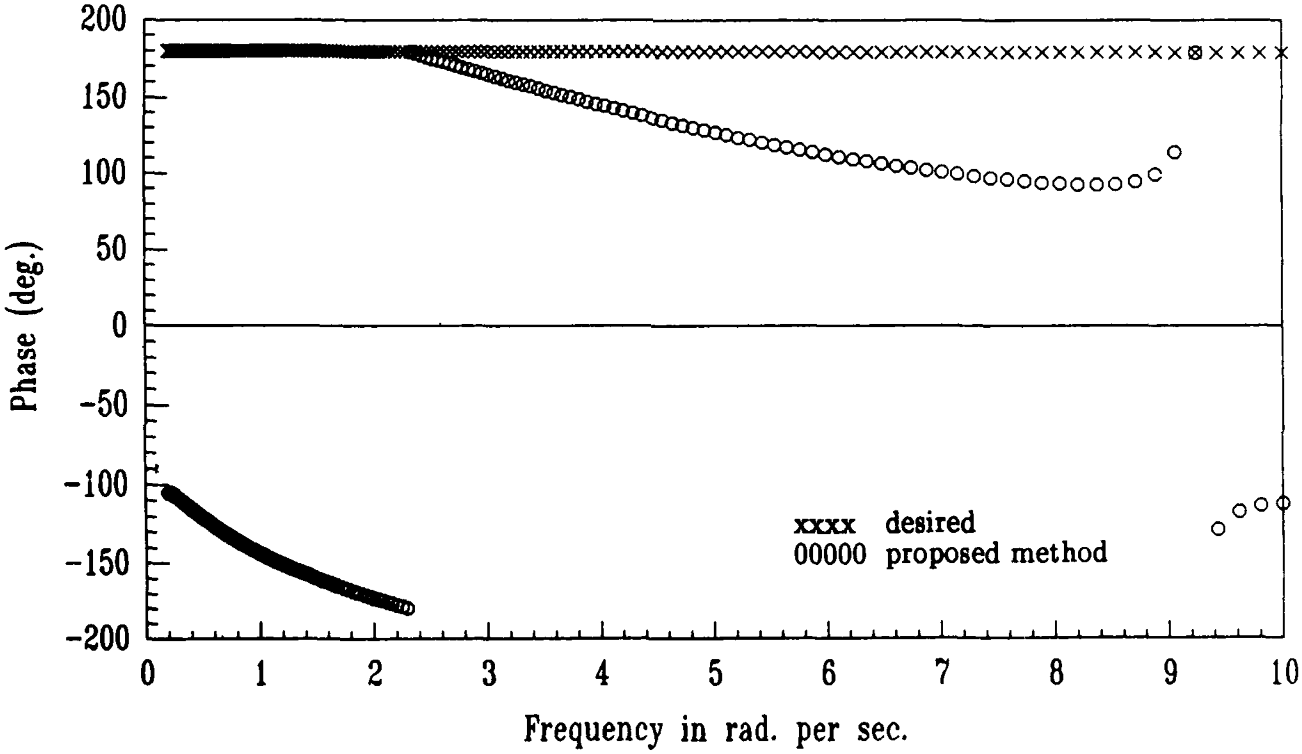

Figure 4.54: Phase response of the (2,1) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, CASE 3.

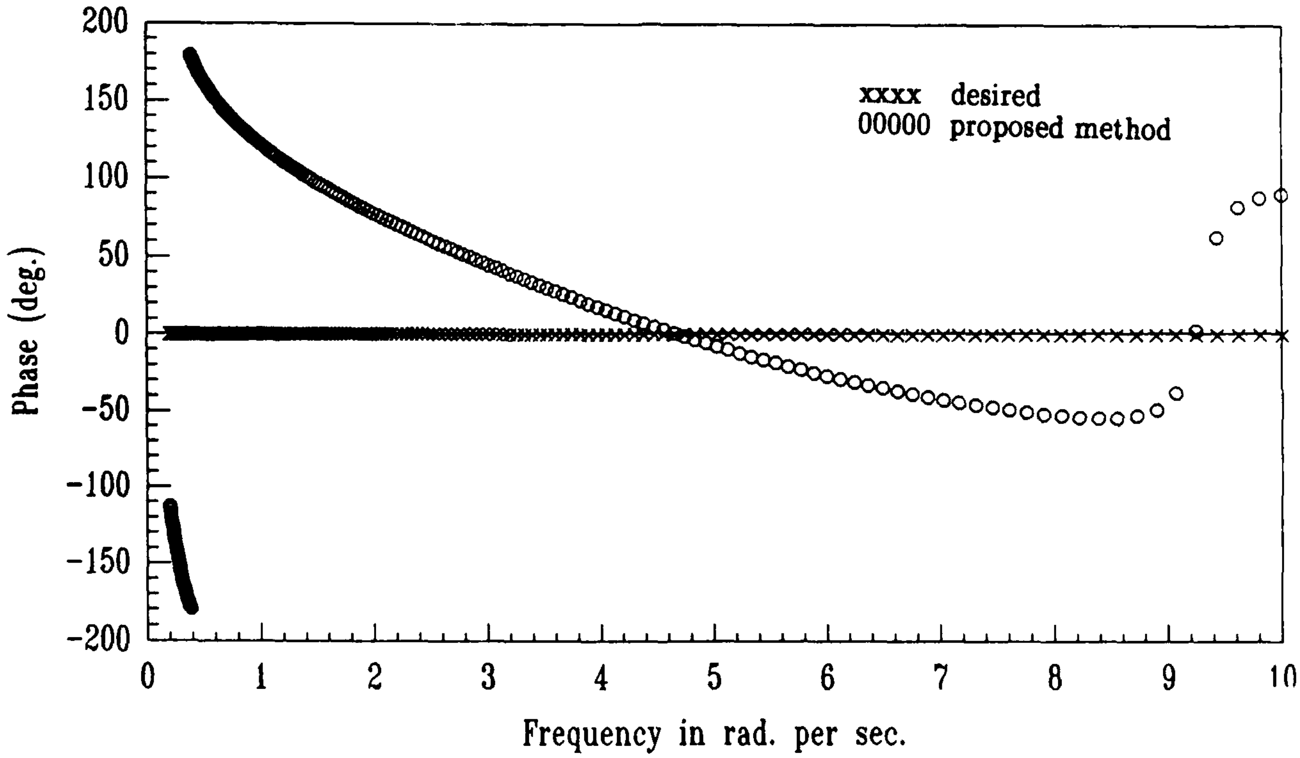

Figure 4.55: Phase response of the (2,2) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, CASE 3.

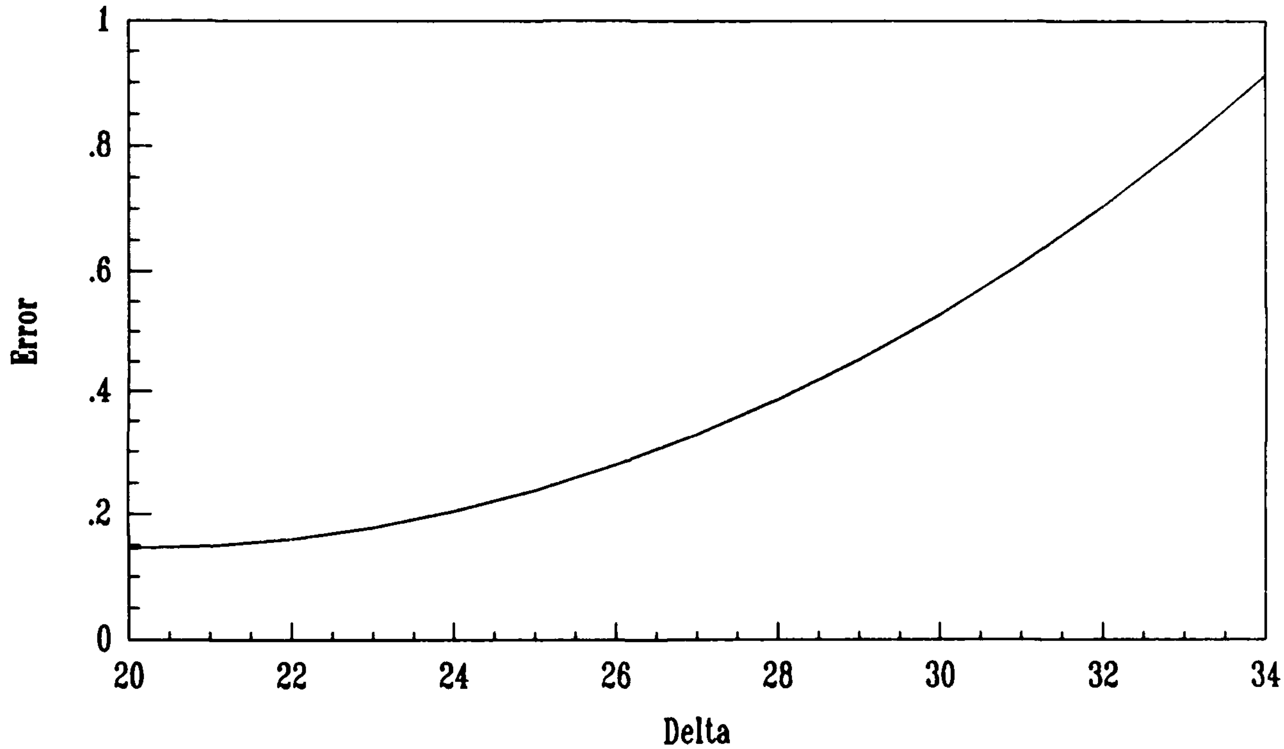

Figure 5.1: Error versus Delta.

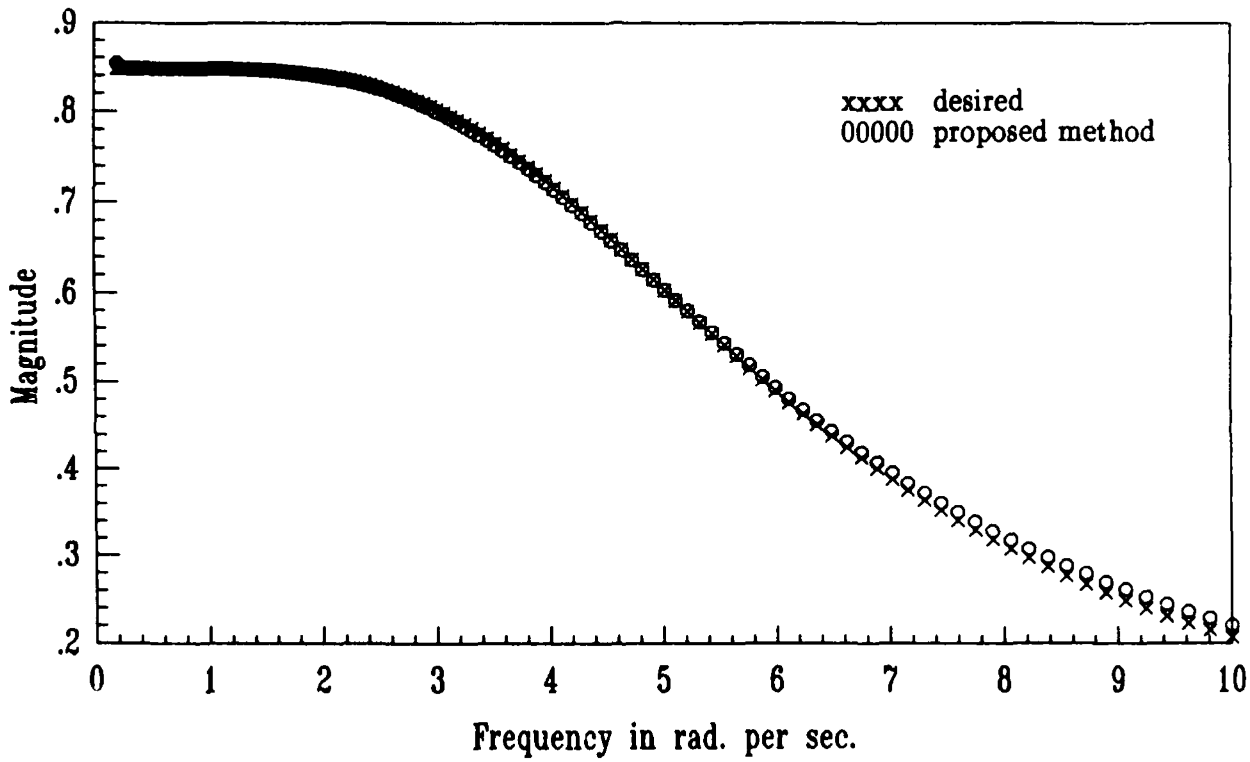

Figure 5.2: Linear magnitude response of the (1,1) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, DELTA = 20.

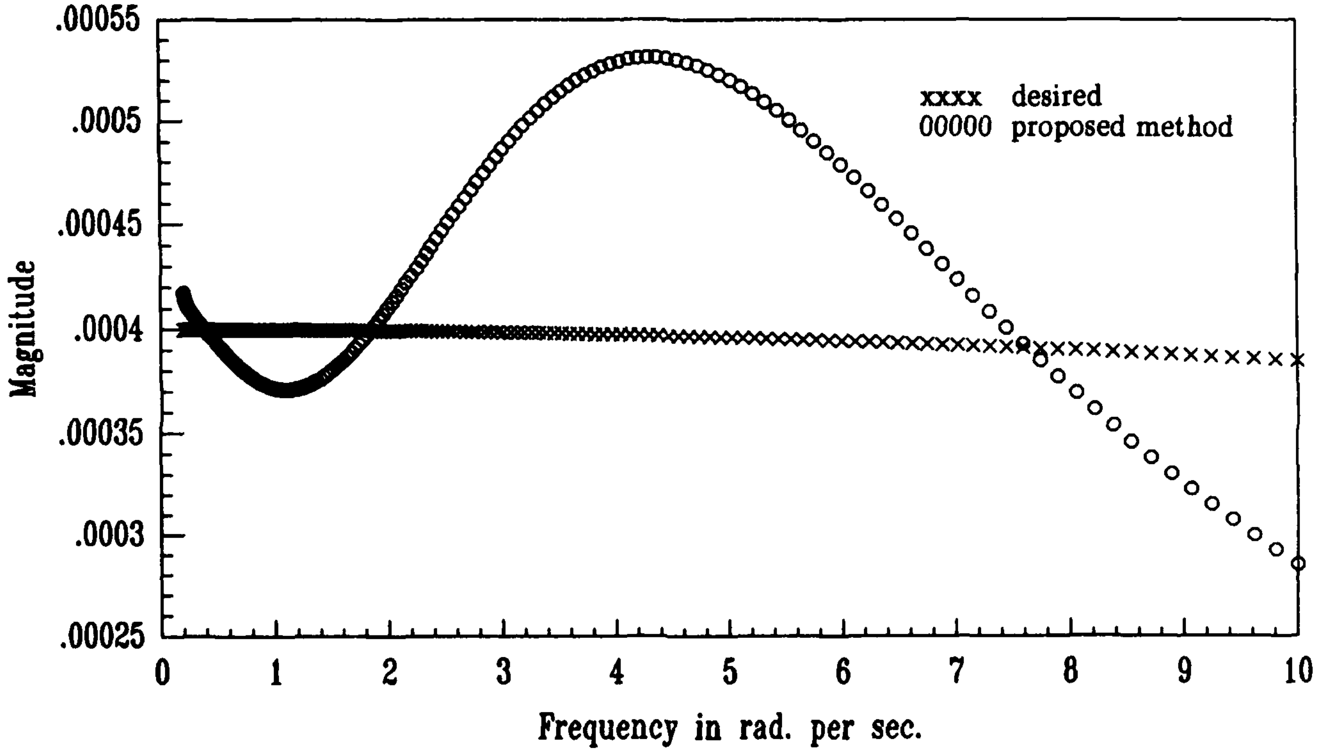

Figure 5.3: Linear magnitude response of the (1,2) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, DELTA = 20.

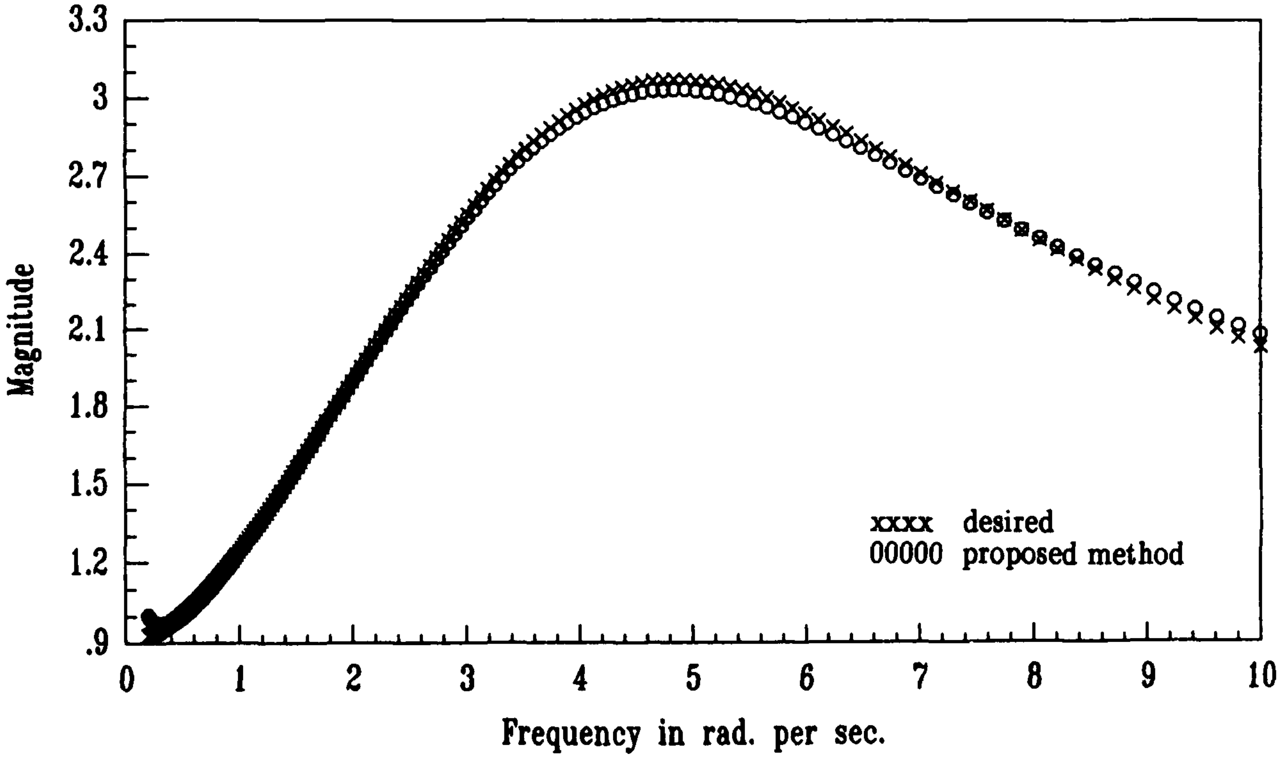

Figure 5.4: Linear magnitude response of the (2,1) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, DELTA = 20.

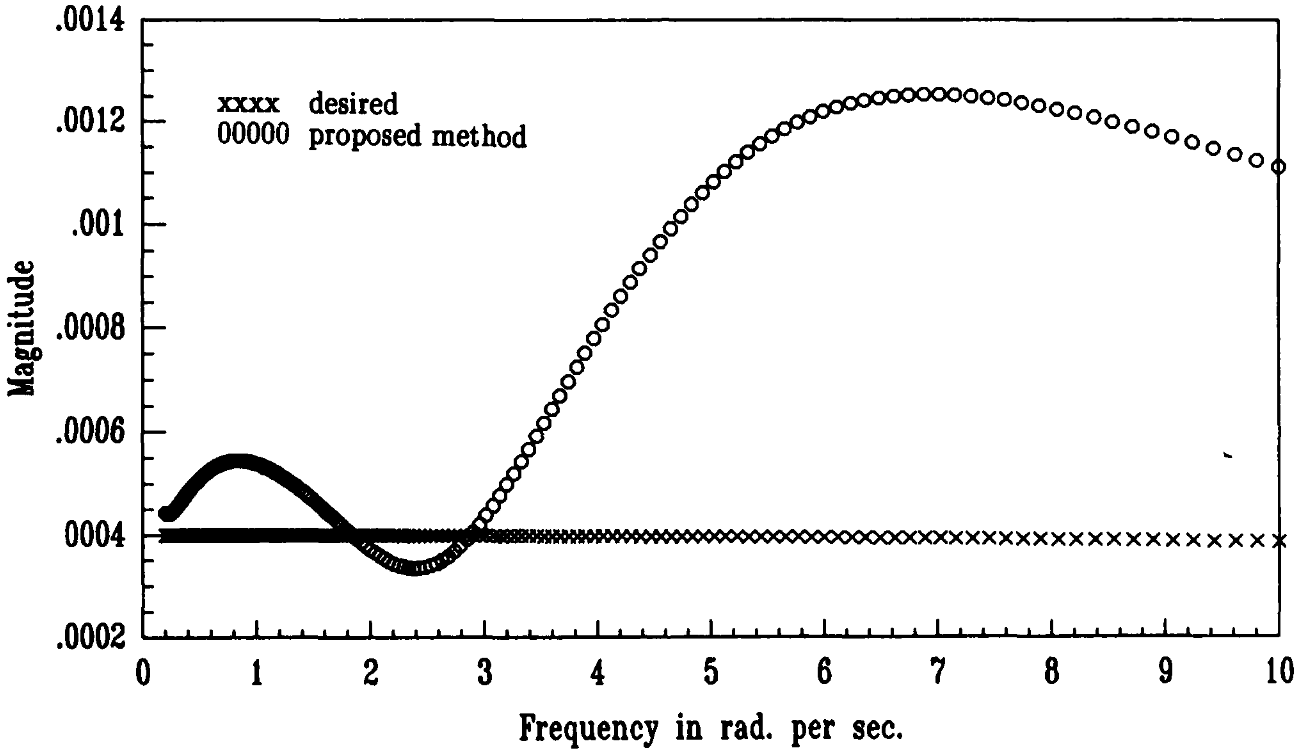

Figure 5.5: Linear magnitude response of the (2,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 20.

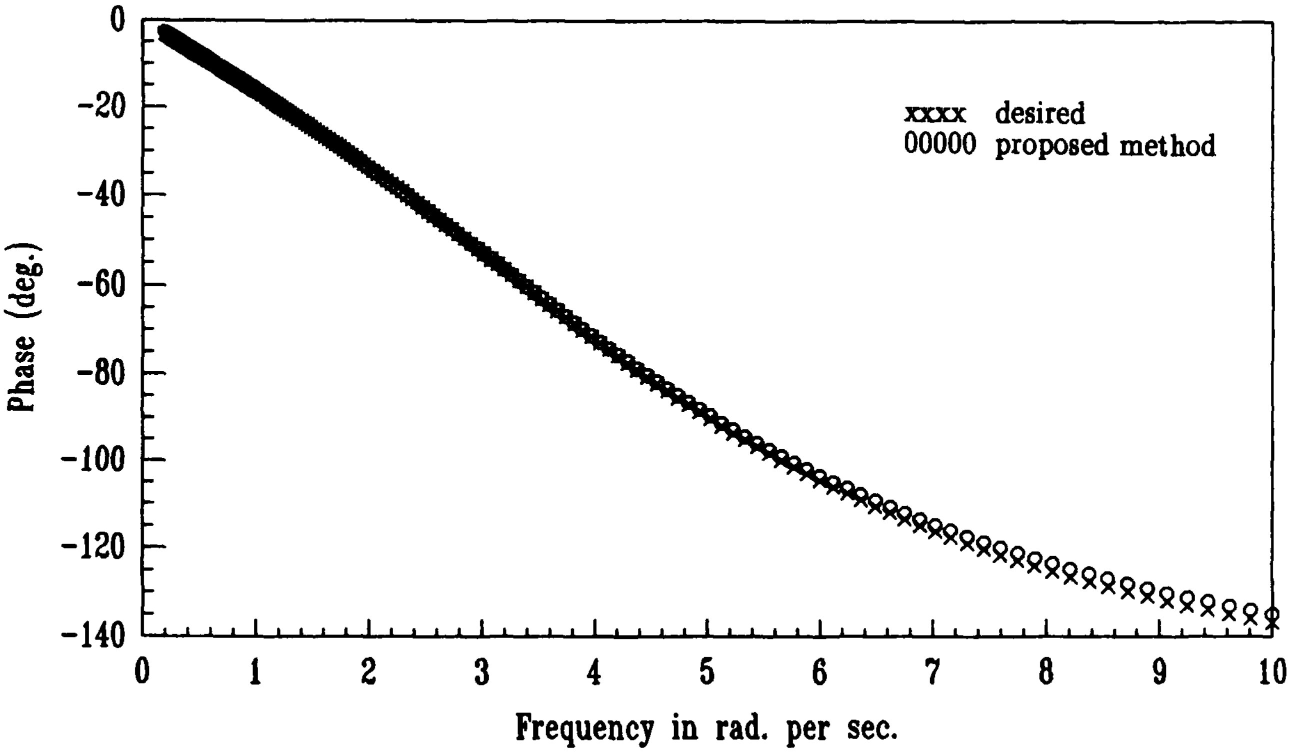

Figure 5.6: Phase response of the (1,1) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, DELTA = 20.

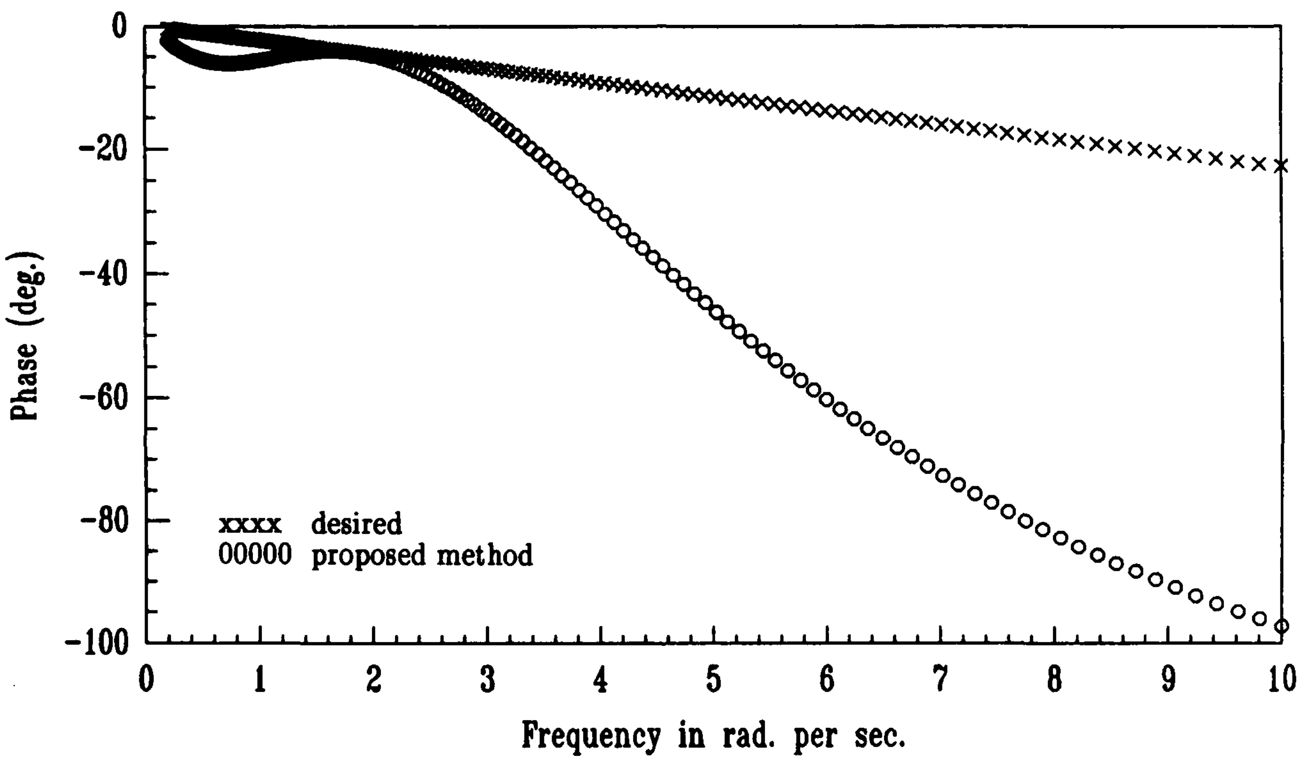

Figure 5.7: Phase response of the (1,2) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, DELTA = 20.

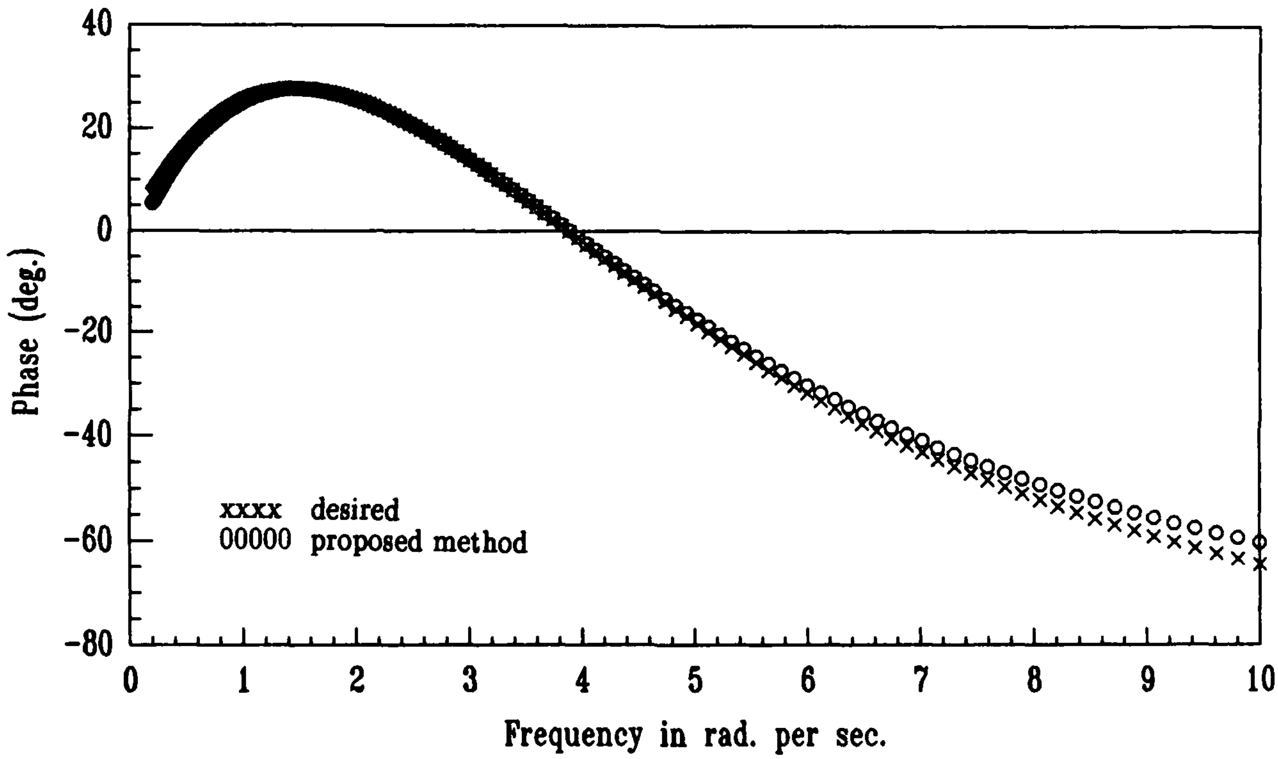

Figure 5.8: Phase response of the (2,1) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 20.

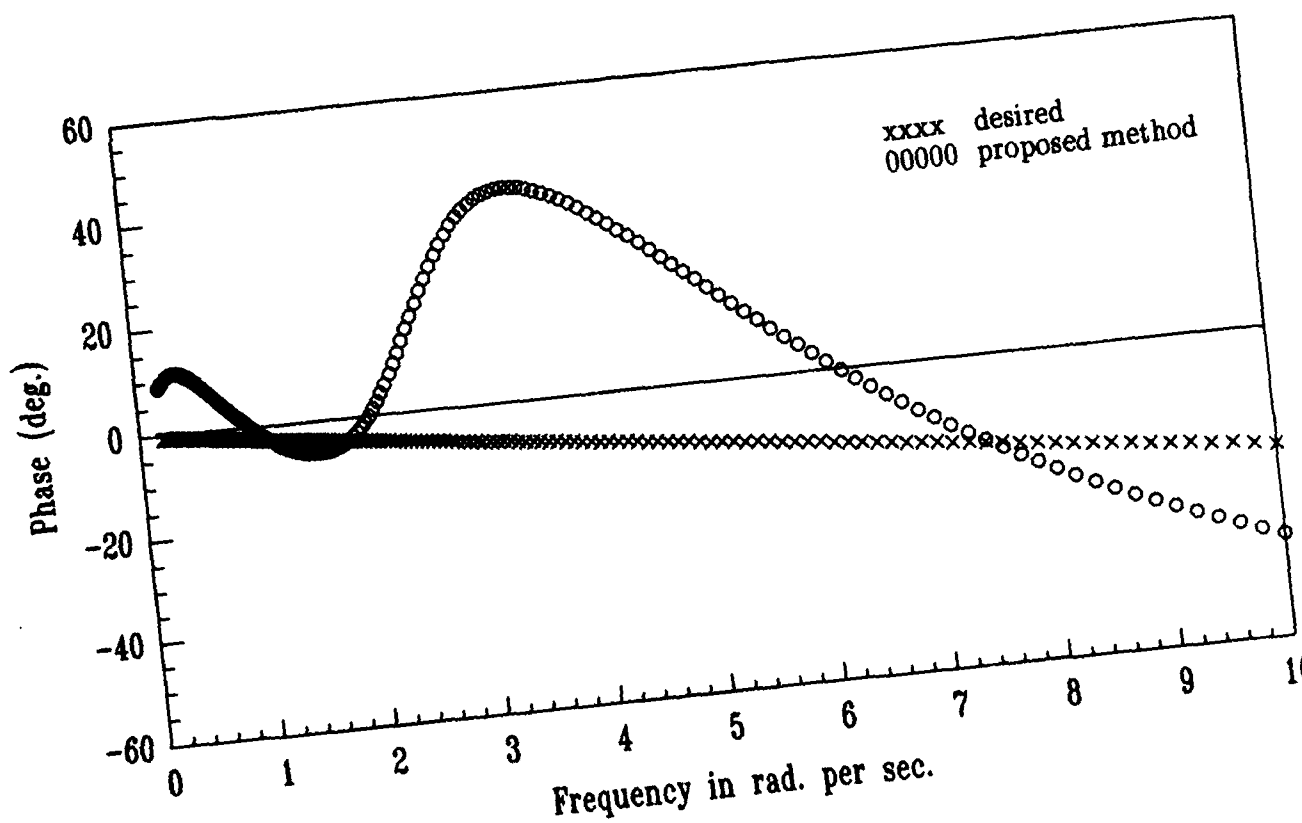

Figure 5.9: Phase response of the (2,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 20.

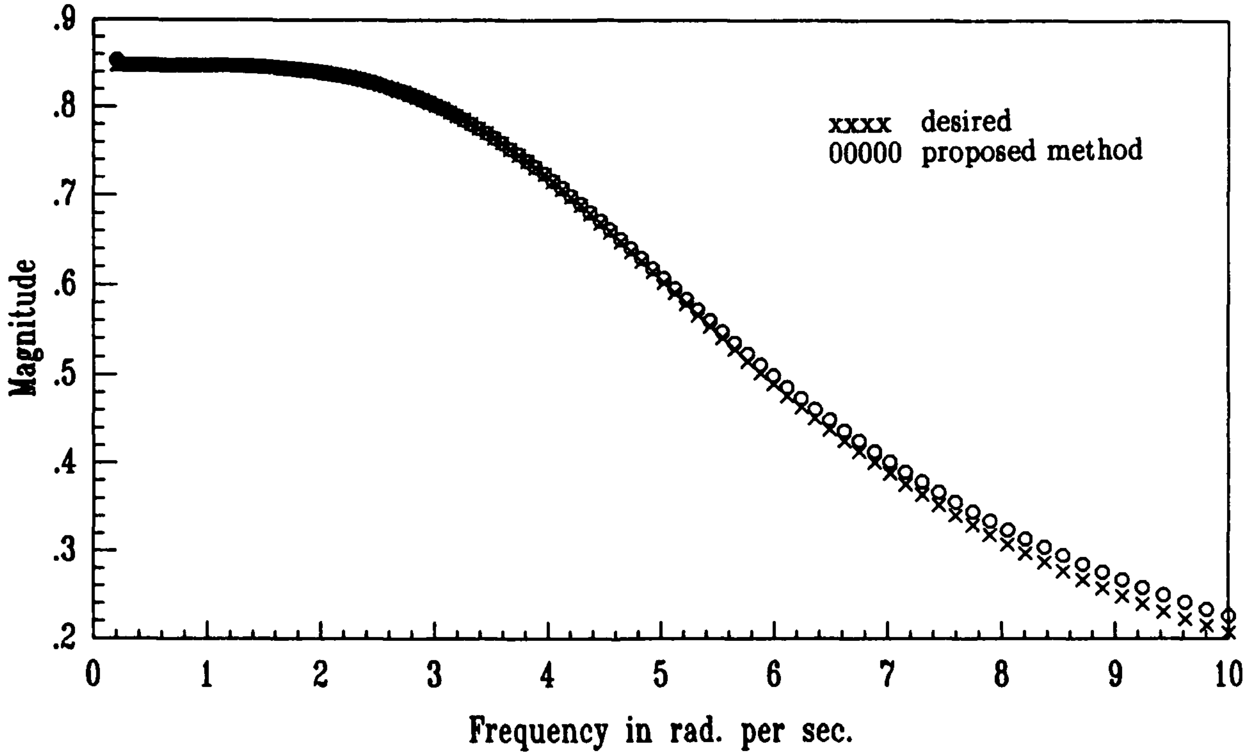

Figure 5.10: Linear magnitude response of the (1,1) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, DELTA = 25.

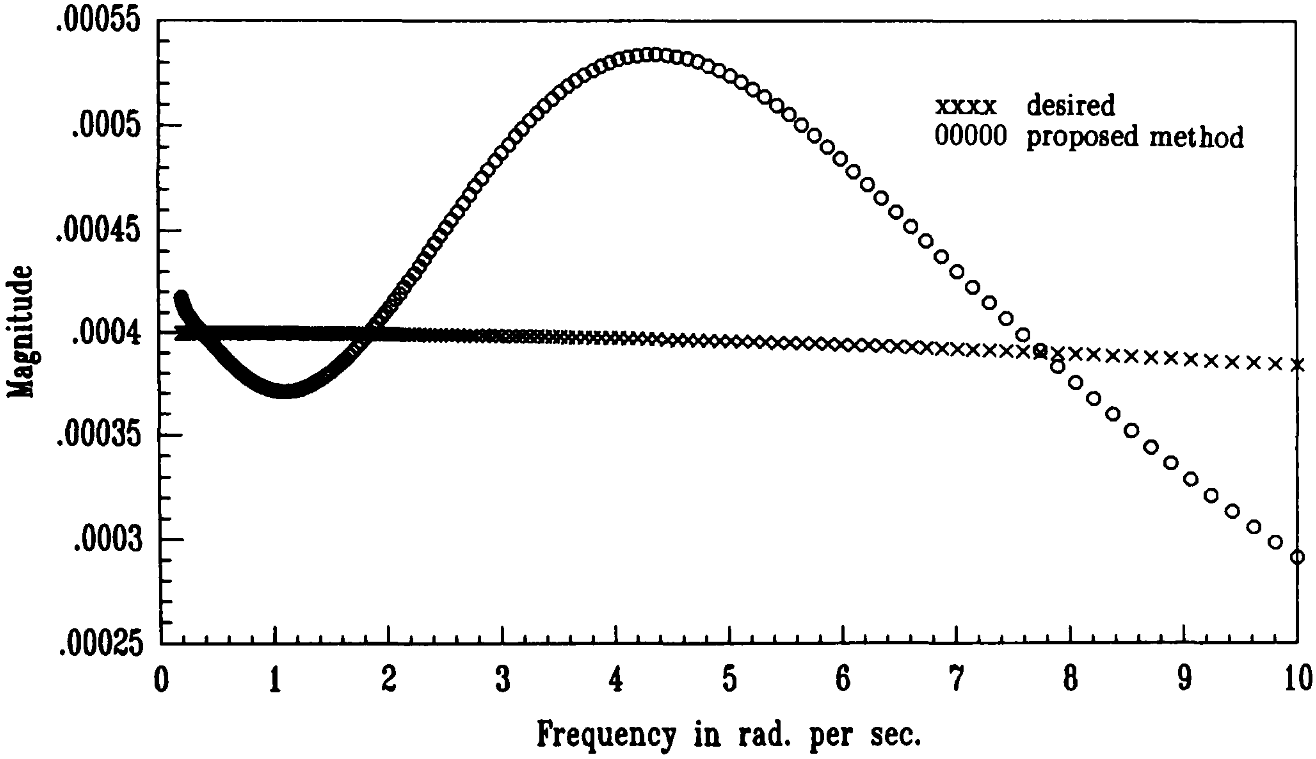

Figure 5.11: Linear magnitude response of the (1,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 25.

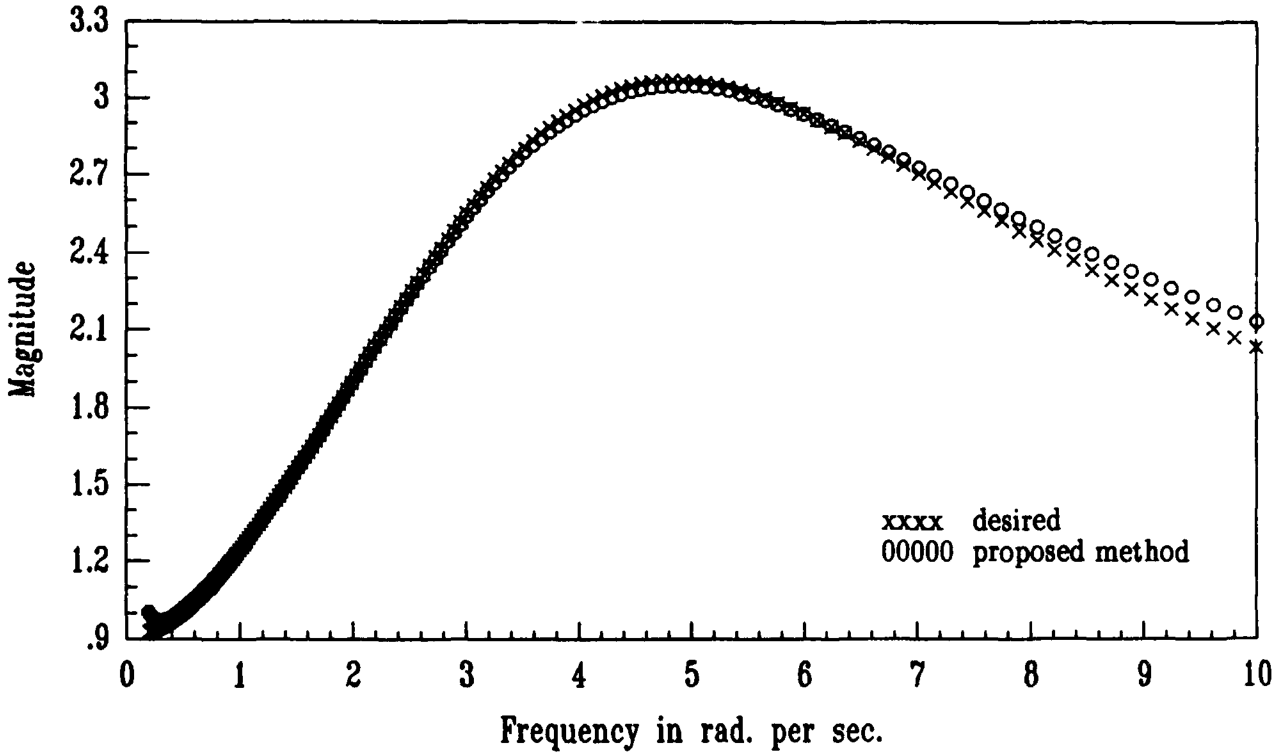

Figure 5.12: Linear magnitude response of the (2,1) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, DELTA = 25.

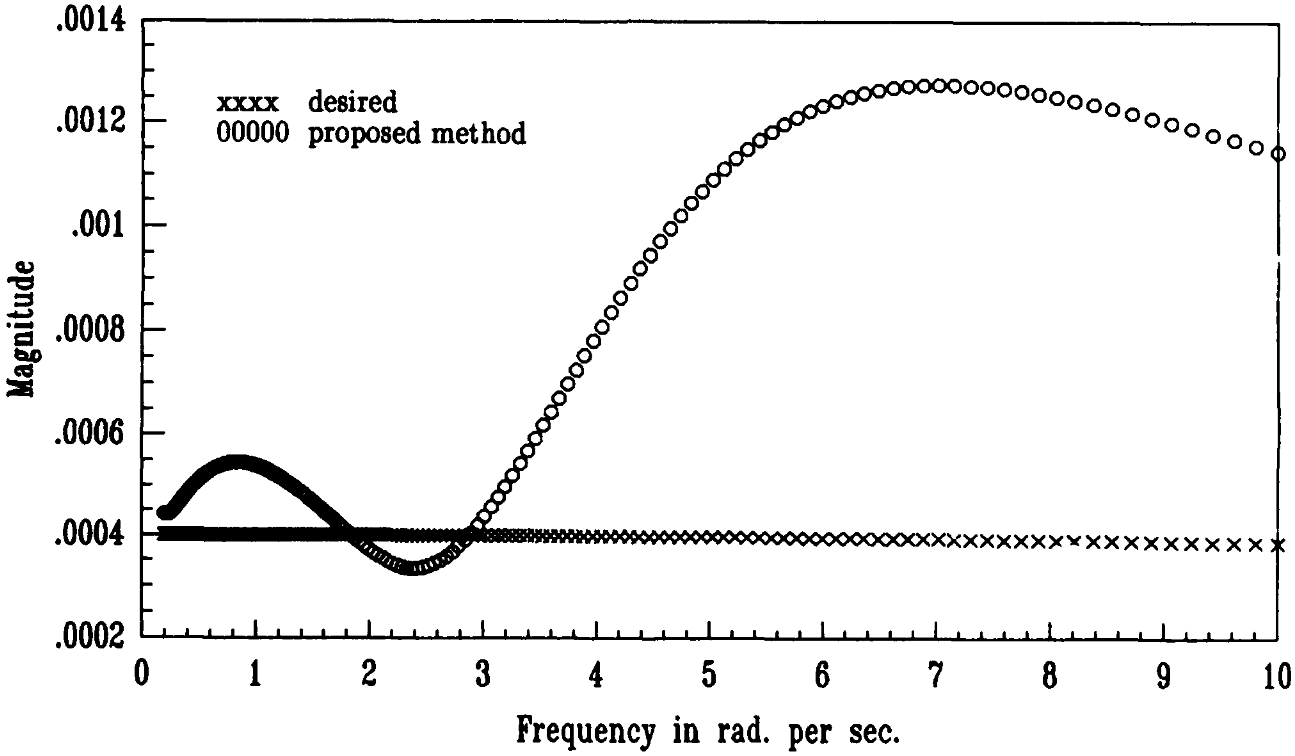

Figure 5.13: Linear magnitude response of the (2,2) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, DELTA = 25.

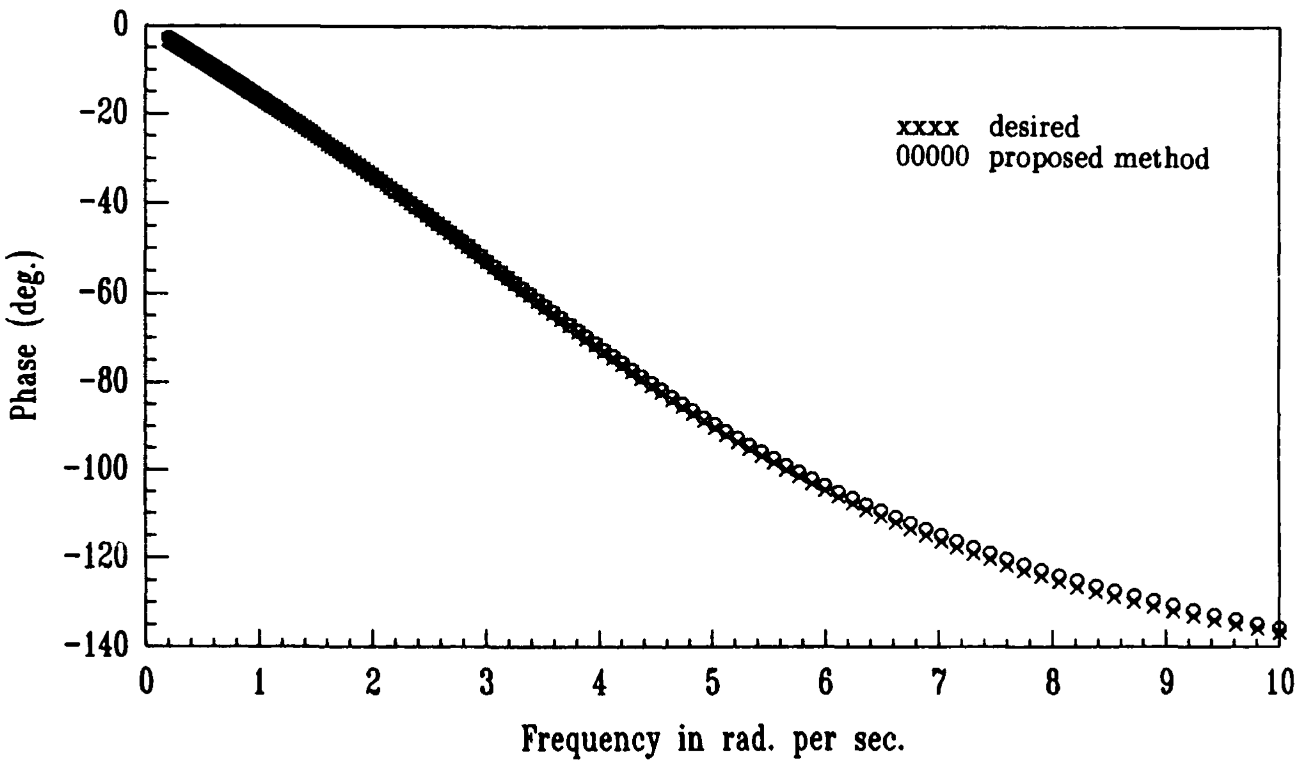

Figure 5.14: Phase response of the (1,1) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, DELTA = 25.

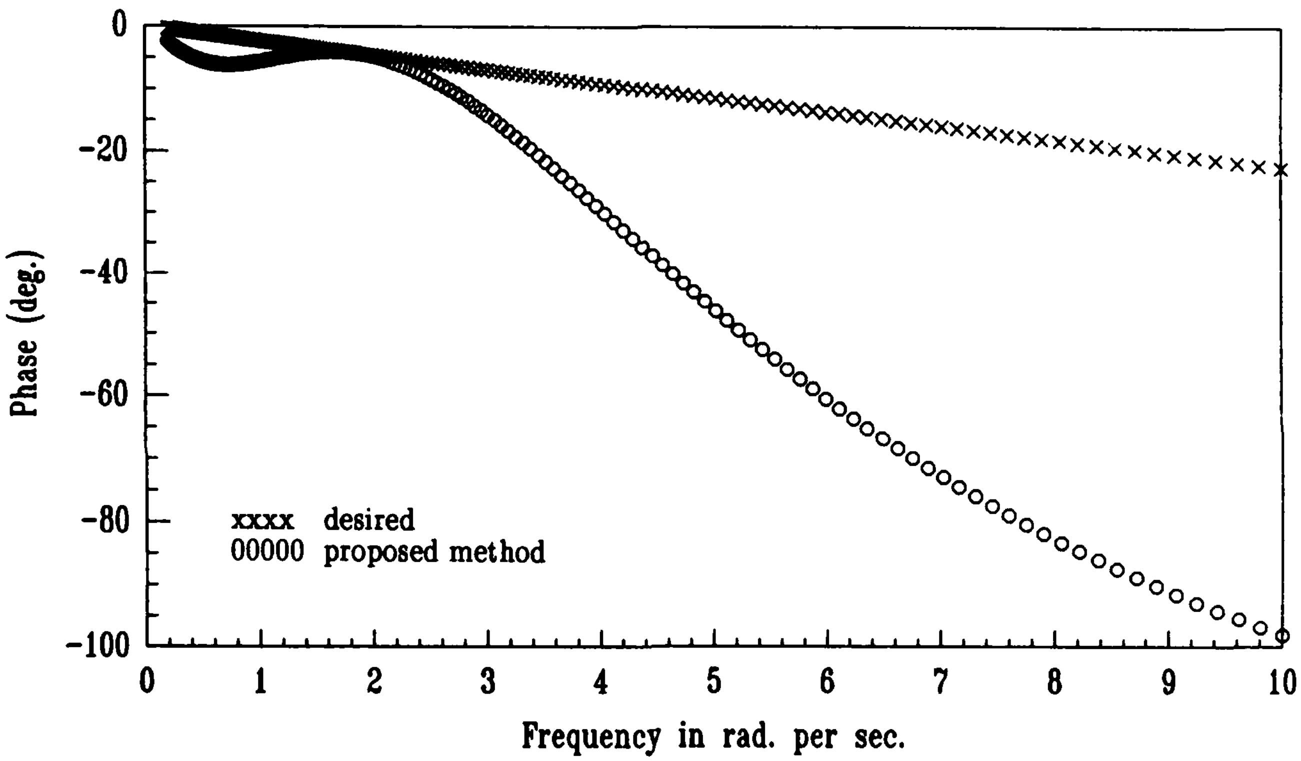

Figure 5.15: Phase response of the (1,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 25.

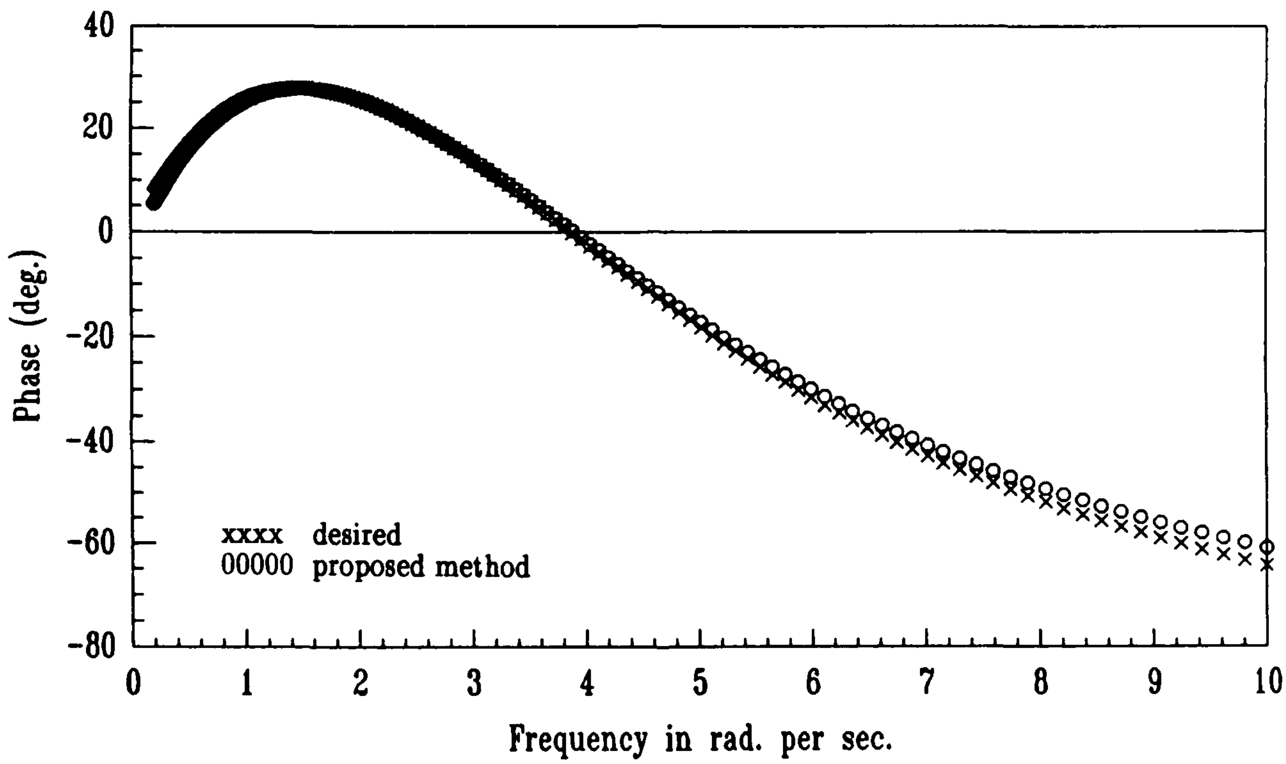

Figure 5.16: Phase response of the (2,1) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, DELTA = 25.

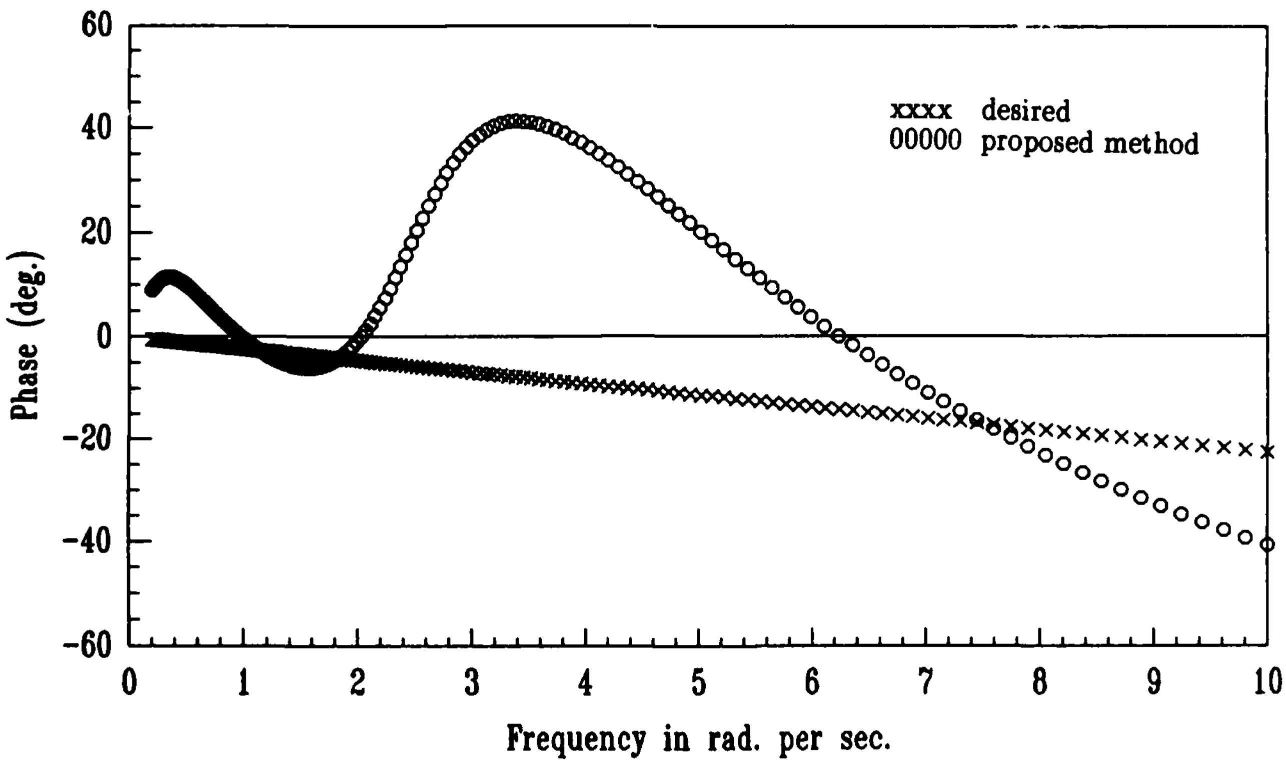

Figure 5.17: Phase response of the (2,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 25.

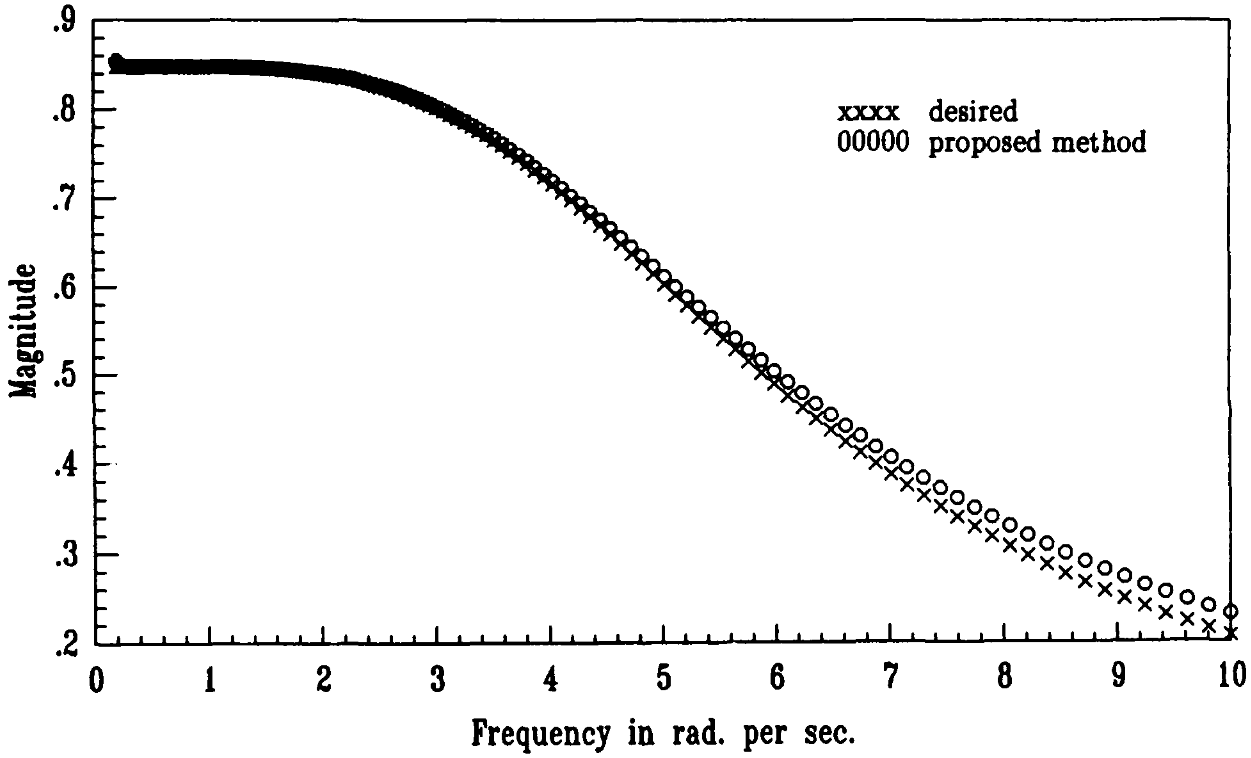

Figure 5.18: Linear magnitude response of the (1,1) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 30.

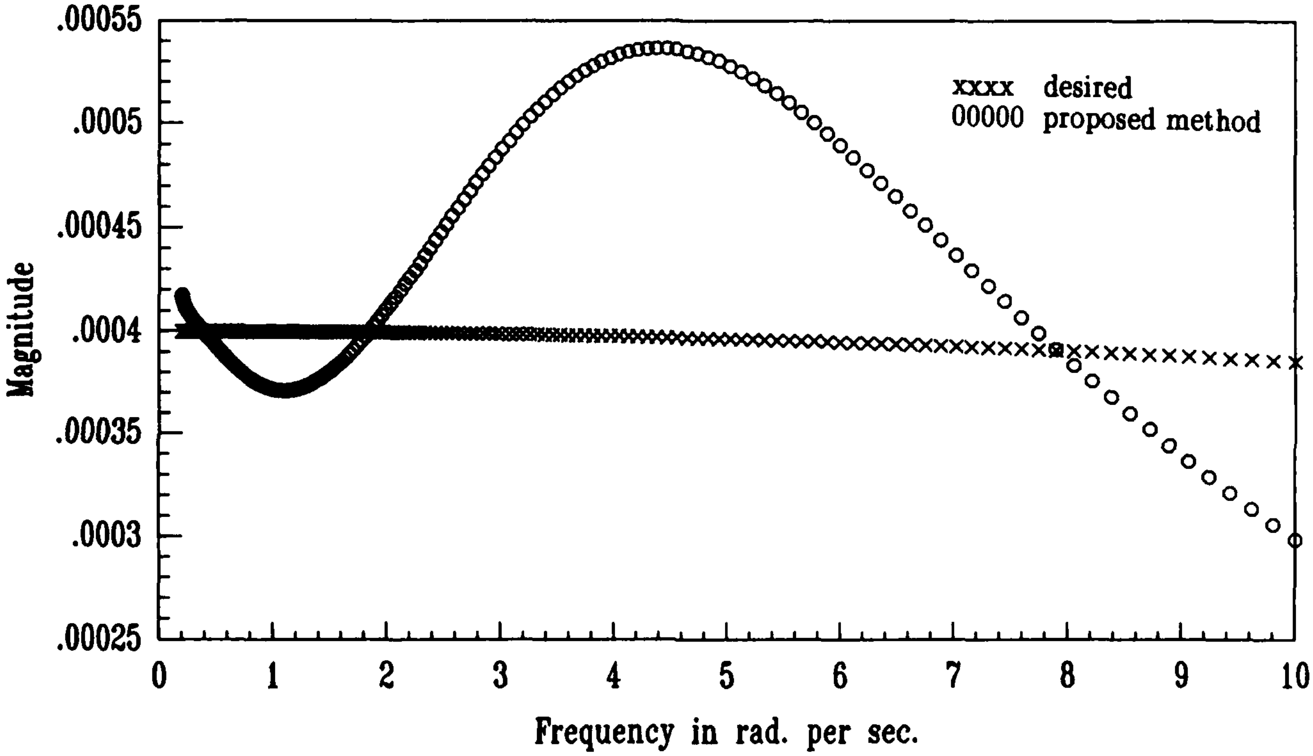

Figure 5.19: Linear magnitude response of the (1,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 30.

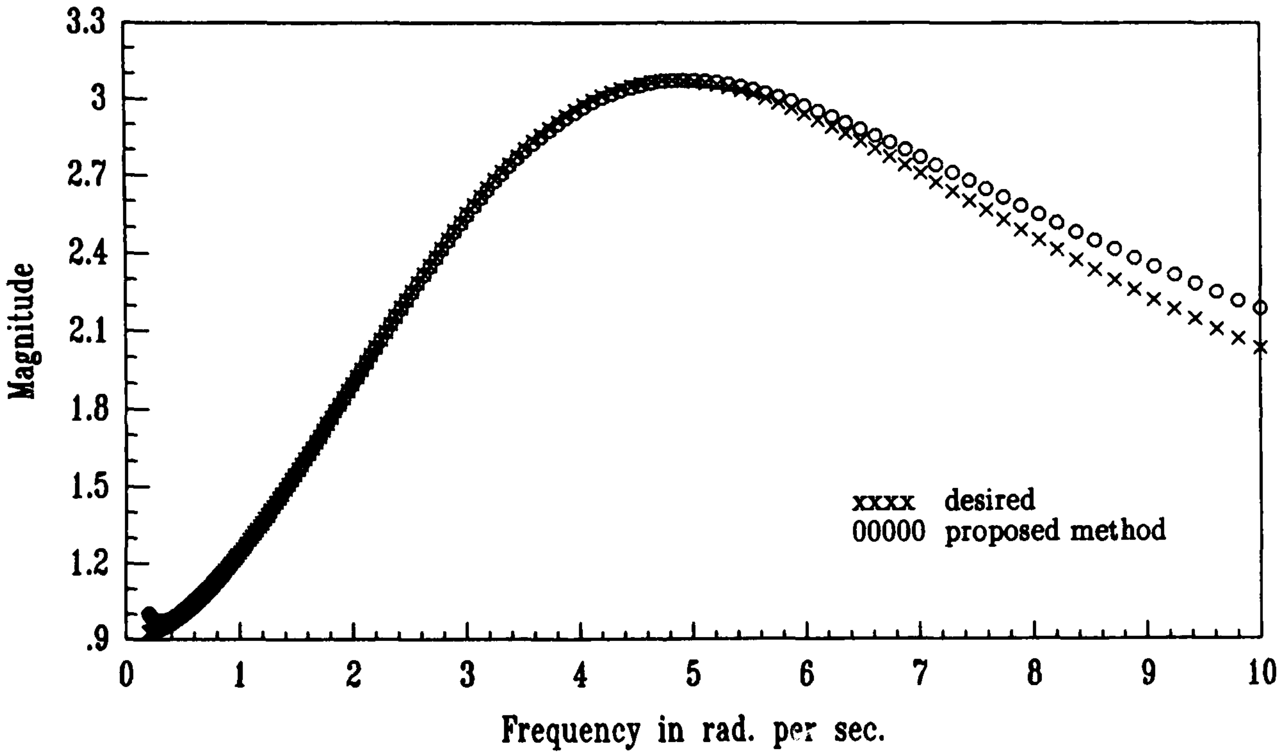

Figure 5.20: Linear magnitude response of the (2,1) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 30.

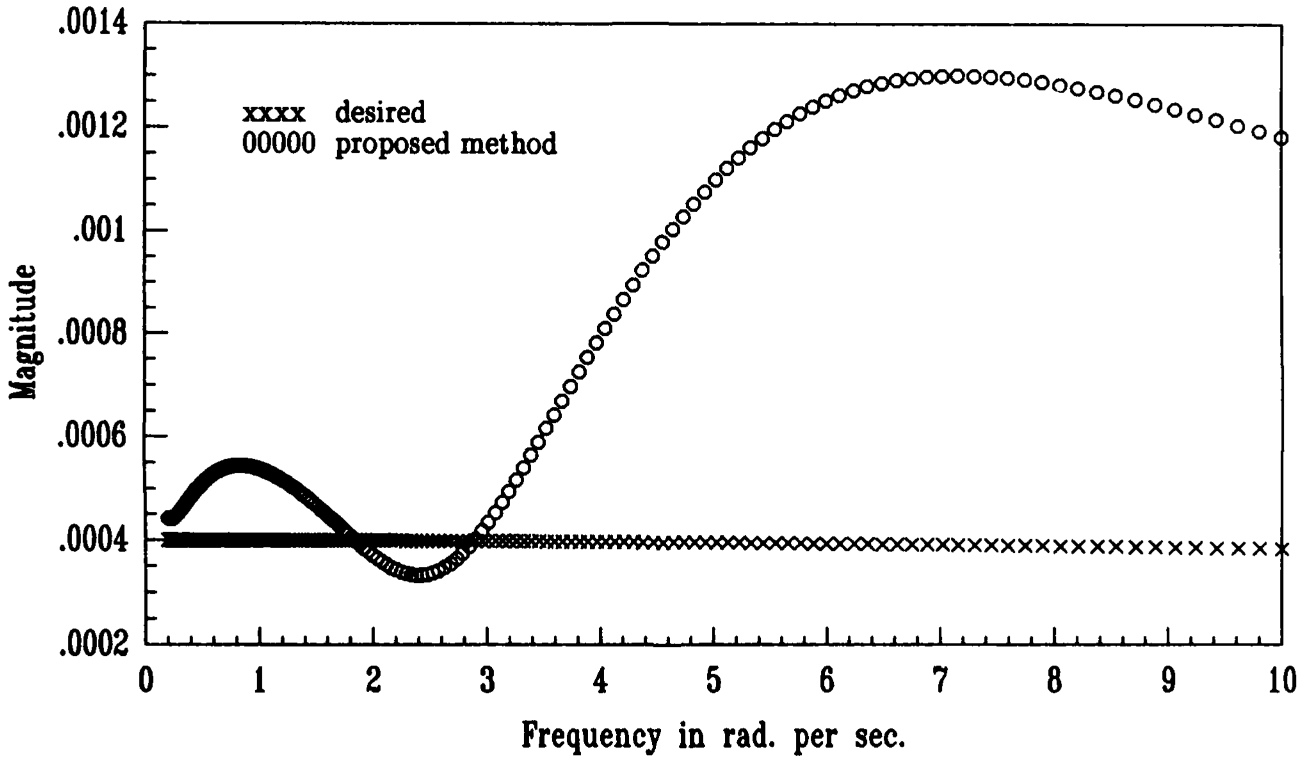

Figure 5.21: Linear magnitude response of the (2,2) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, DELTA = 30.

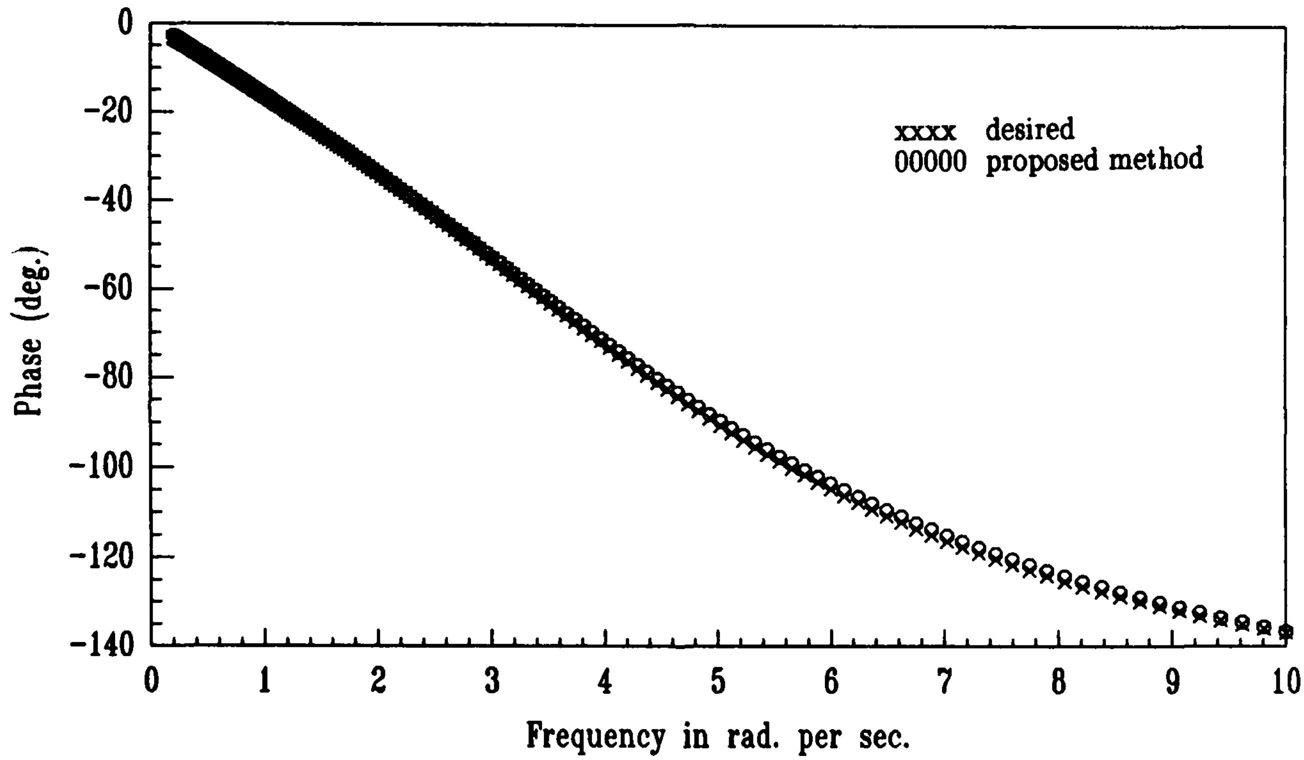

Figure 5.22: Phase response of the (1,1) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, DELTA = 30.

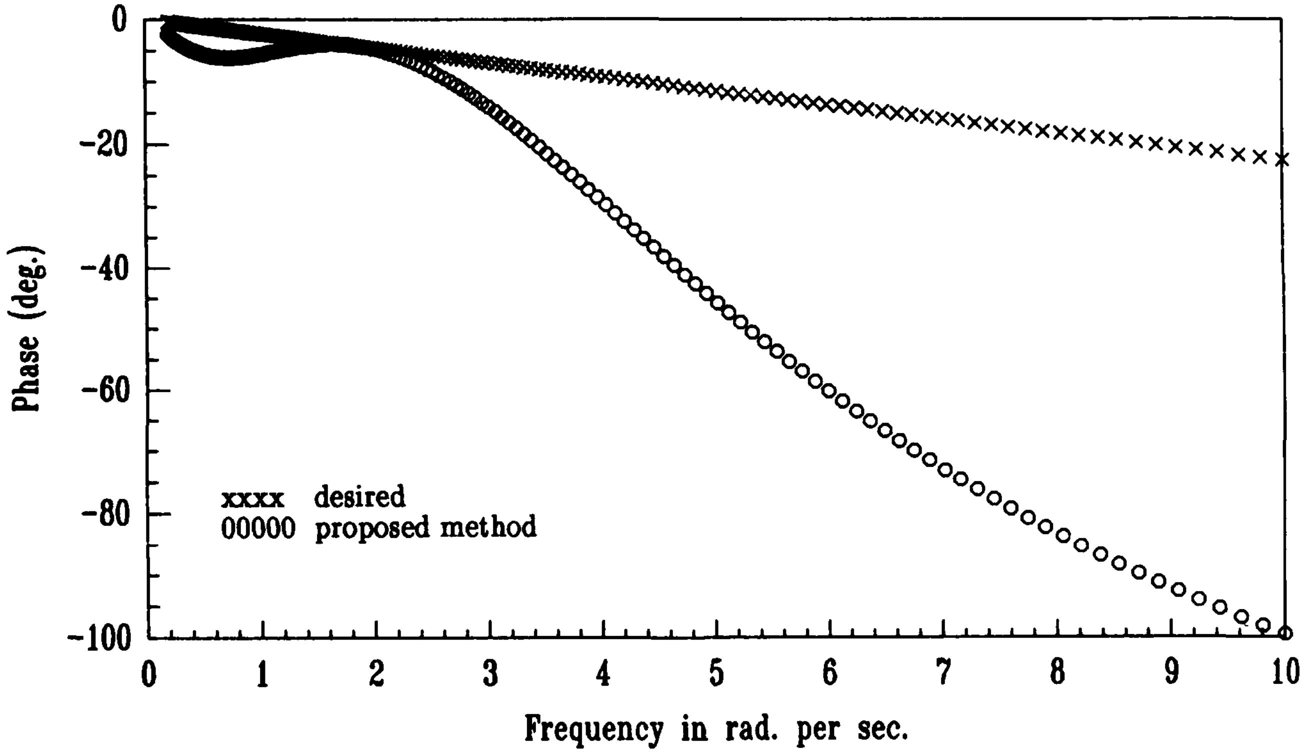

Figure 5.23: Phase response of the (1,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 30.

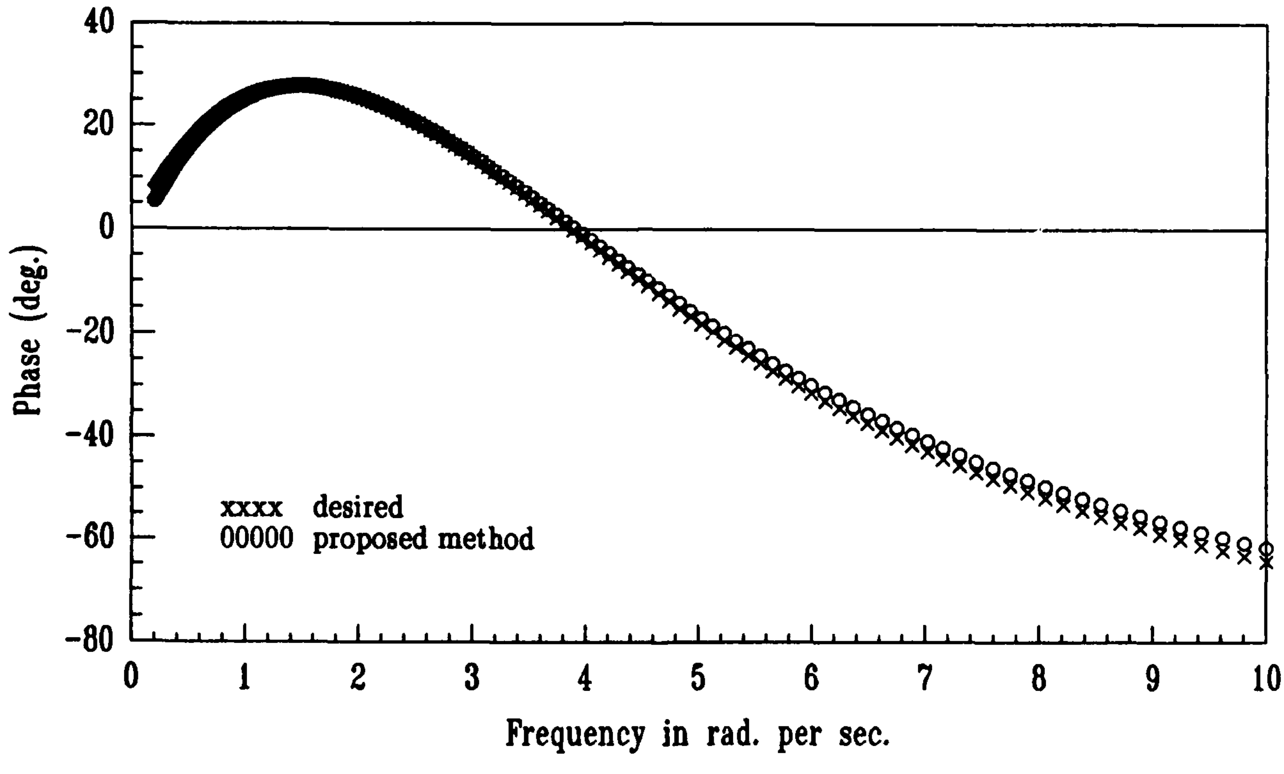

Figure 5.24: Phase response of the (2,1) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 30.

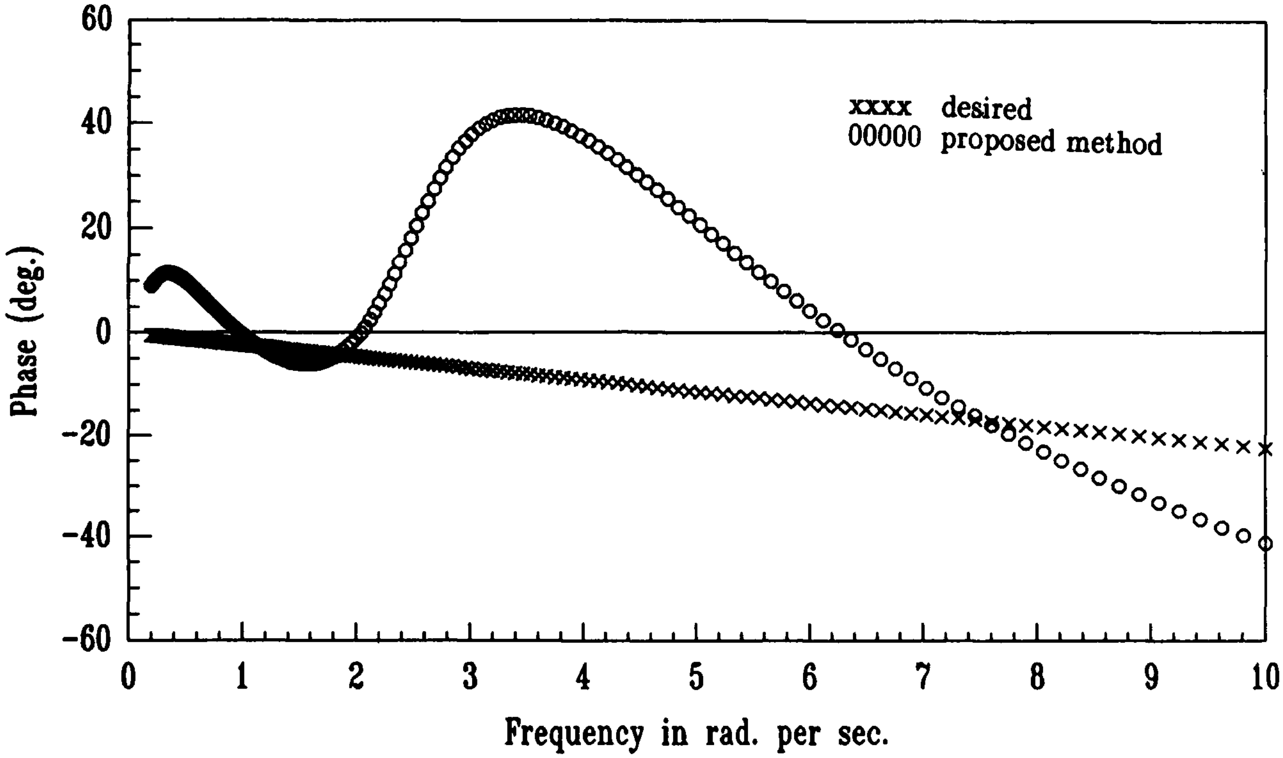

Figure 5.25: Phase response of the (2,2) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, DELTA = 30.

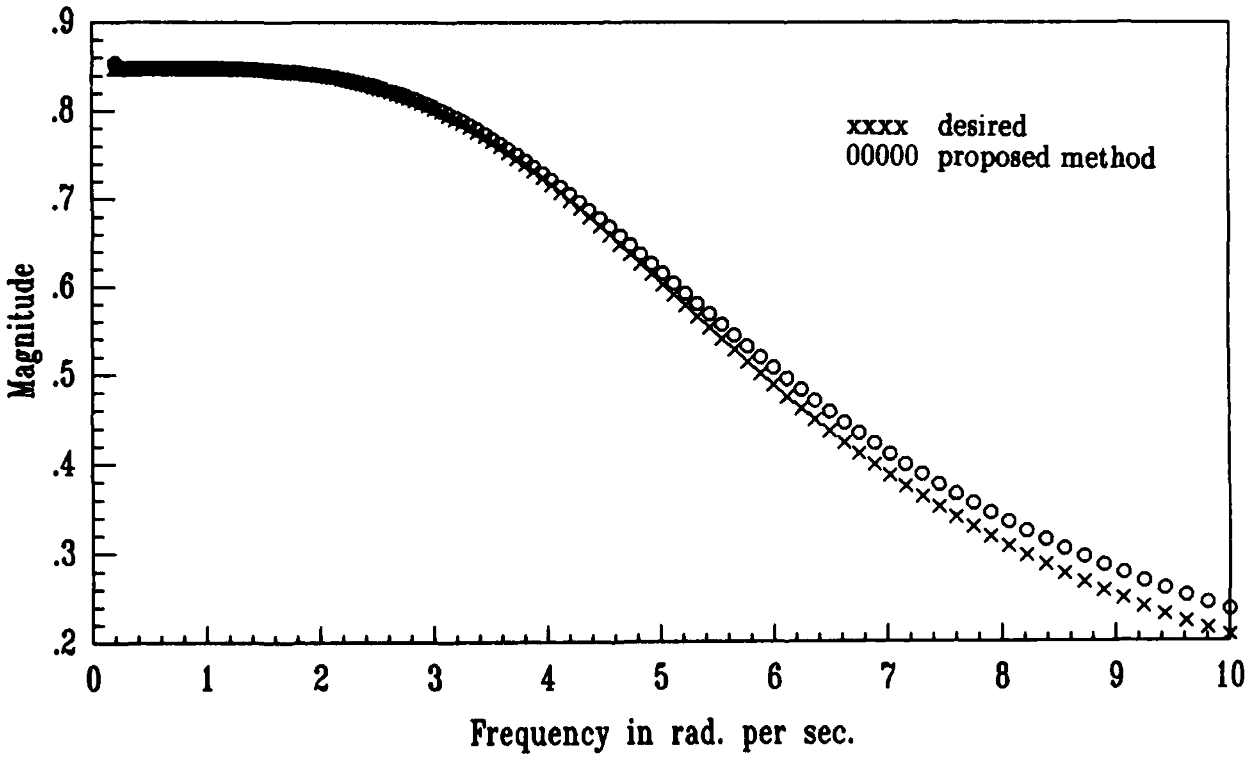

Figure 5.26: Linear magnitude response of the (1,1) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, DELTA = 34.

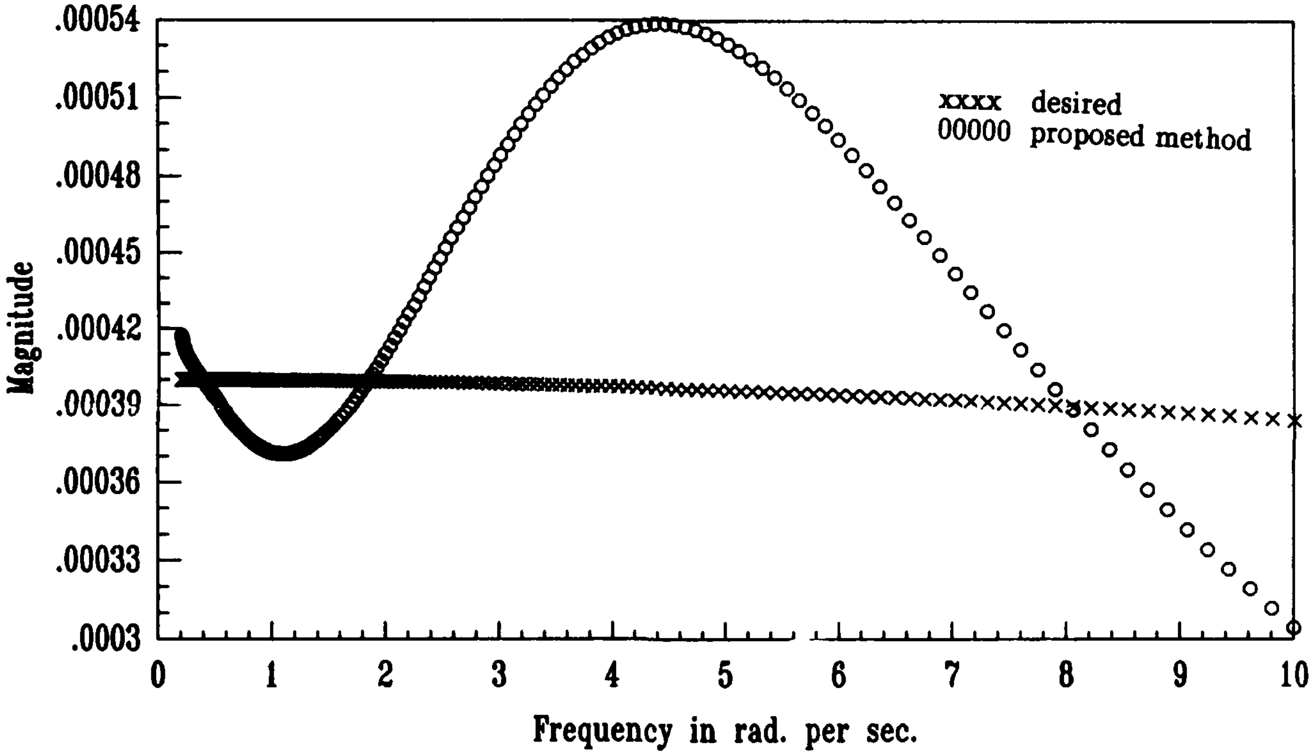

Figure 5.27: Linear magnitude response of the (1,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 34.

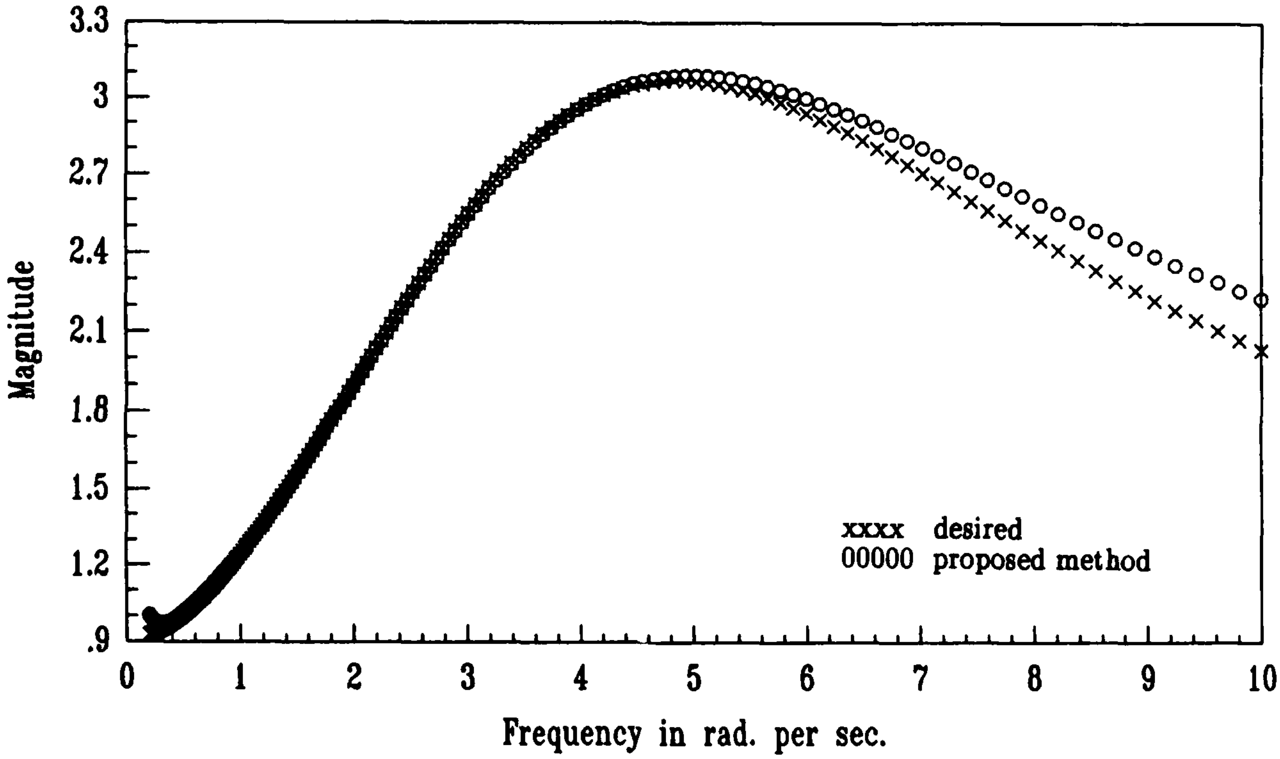

Figure 5.28: Linear magnitude response of the (2,1) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 34.

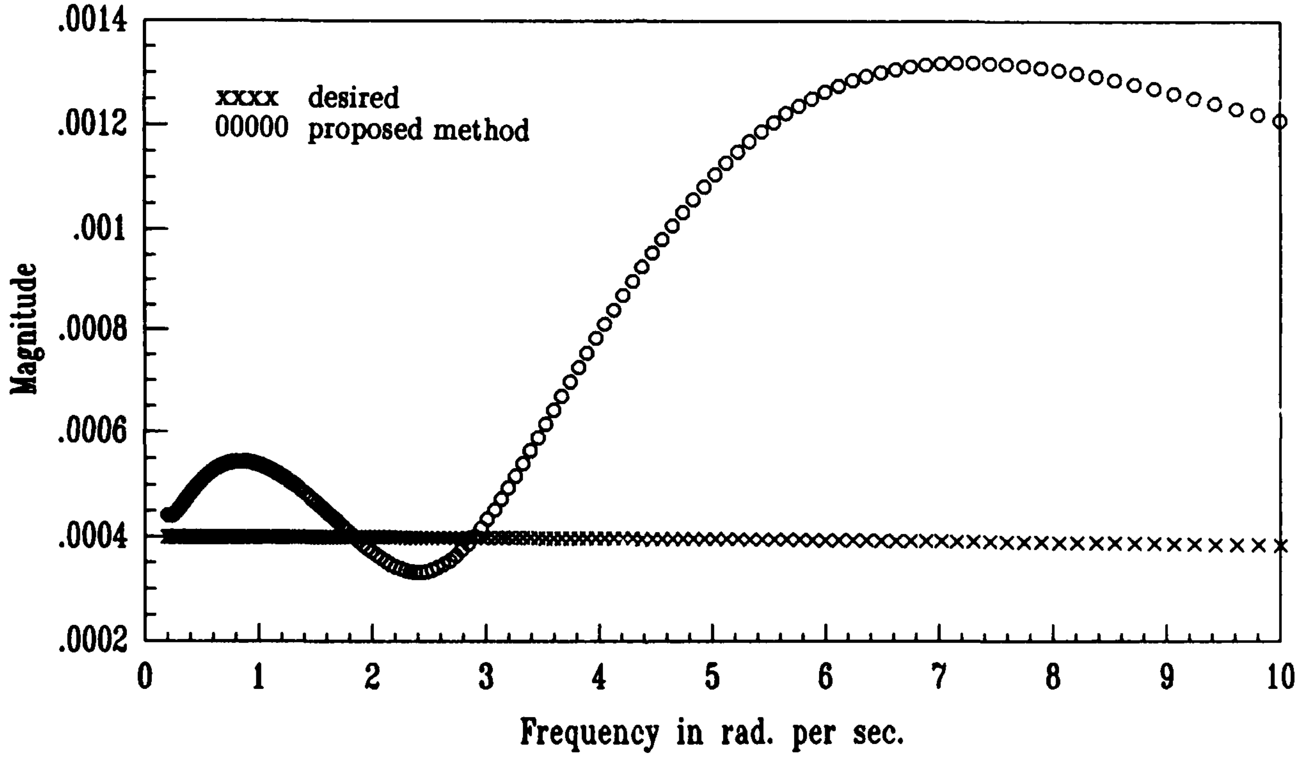

Figure 5.29: Linear magnitude response of the (2,2) element of the compensated and 'desired' closed loop transfer function matrix for YF–16 CCV, DELTA = 34.

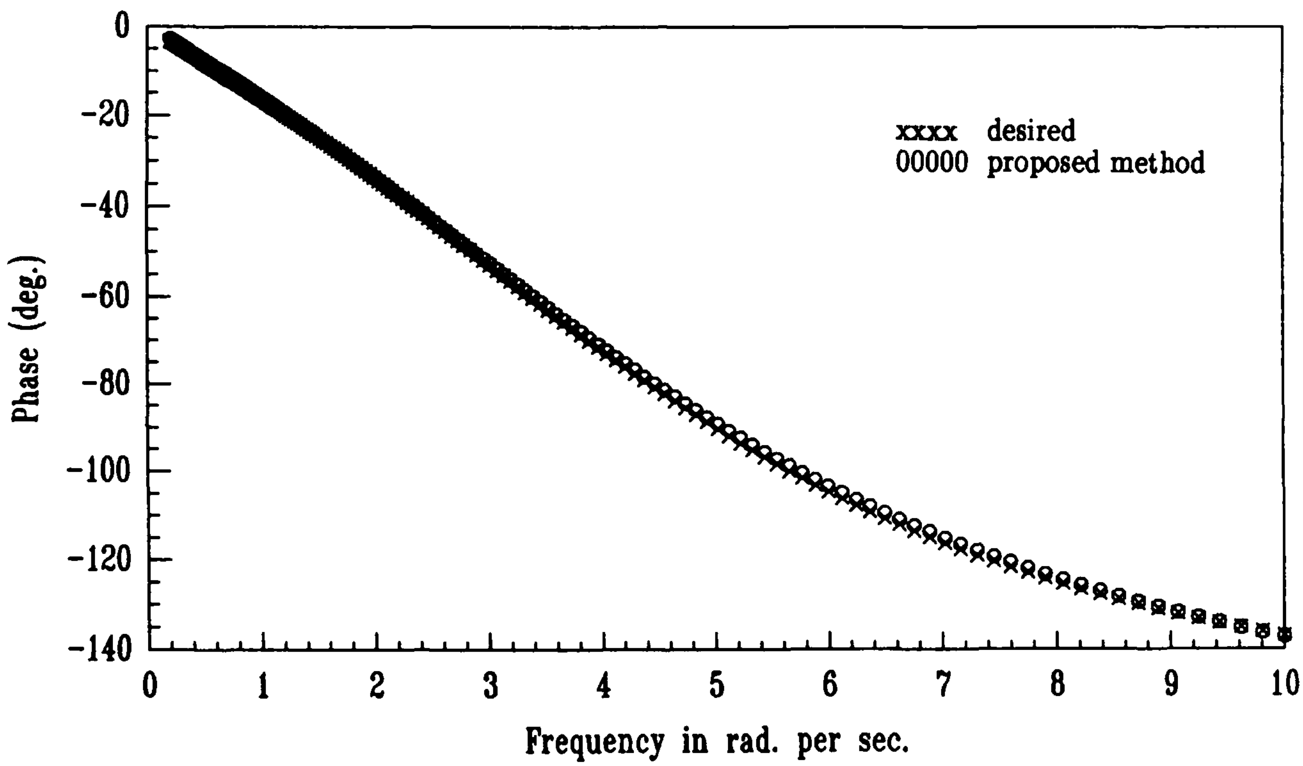

Figure 5.30: Phase response of the (1,1) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 34.

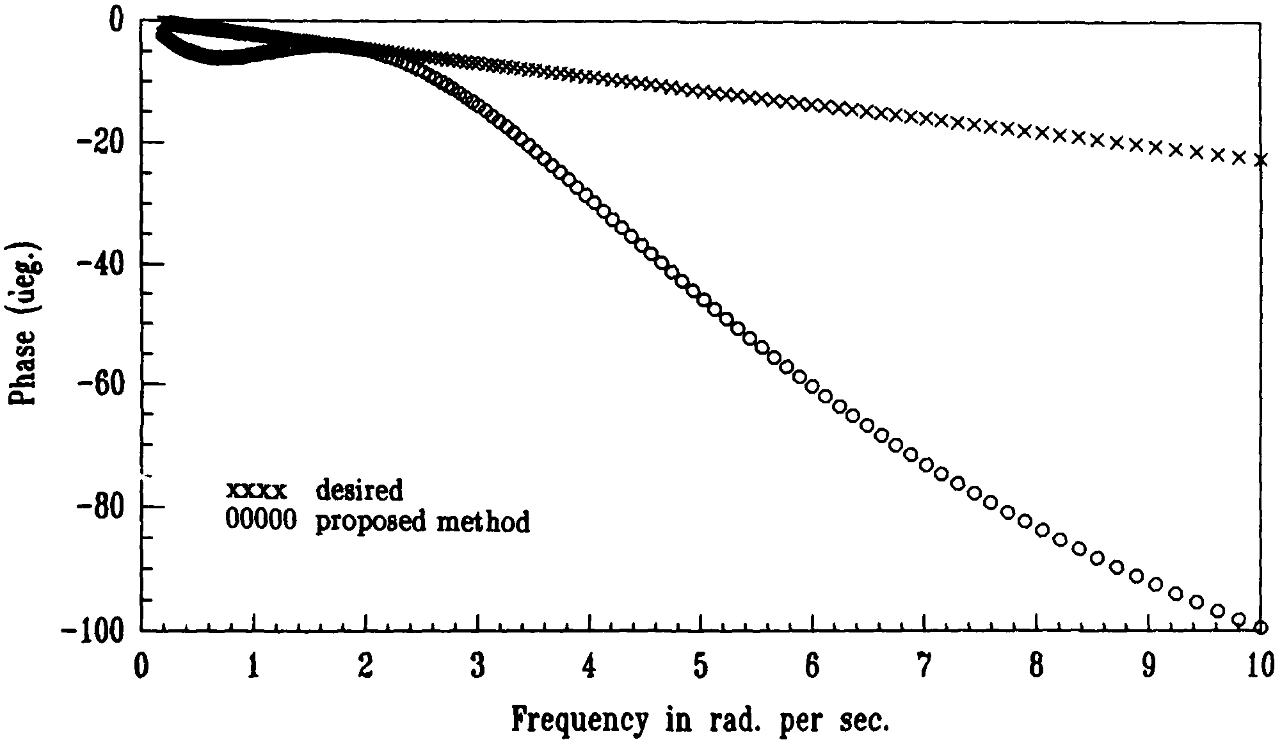

Figure 5.31: Phase response of the (1,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 34.

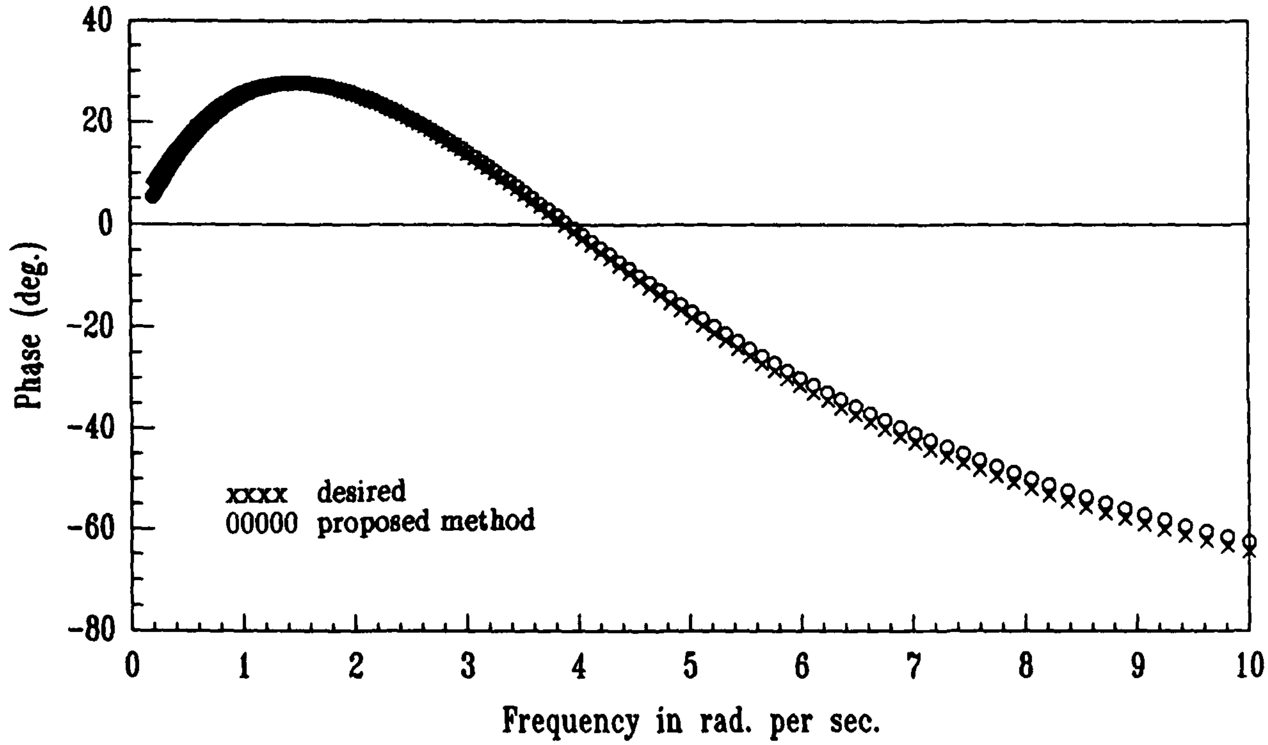

Figure 5.32: Phase response of the (2,1) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 34.

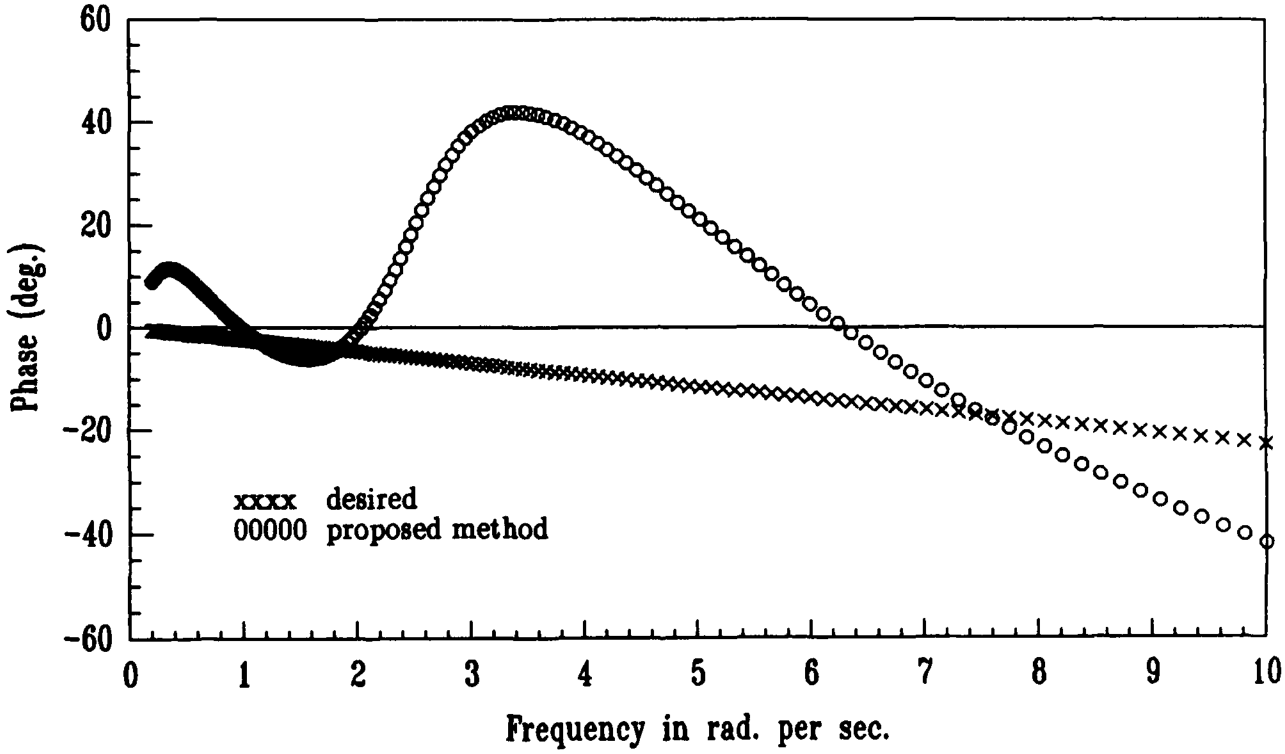

Figure 5.33: Phase response of the (2,2) element of the compensated and 'desired' closed loop transfer function matrix for YF-16 CCV, DELTA = 34.

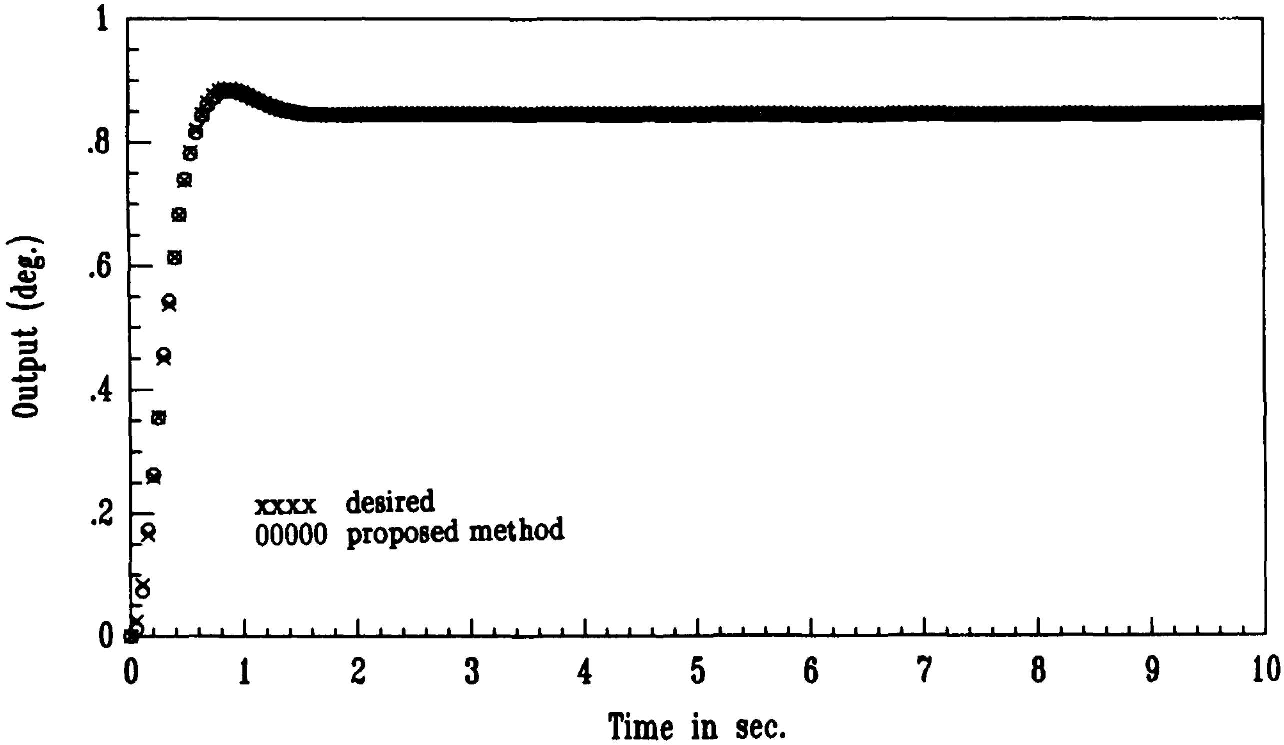

Figure 5.34: Time response of the (1,1) element of the compensated system and the 'desired' system for YF–16 CCV, DELTA = 20.

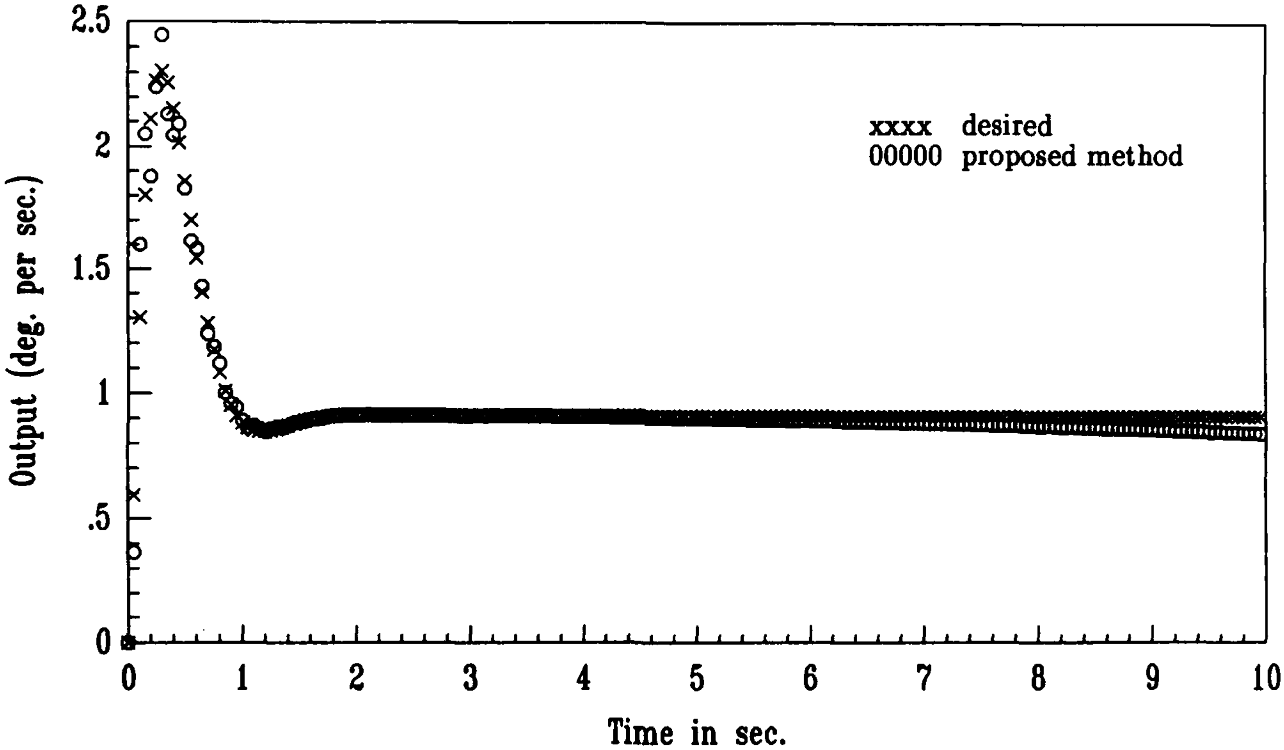

Figure 5.35: Time response of the (1,2) element of the compensated system and the 'desired' system for YF–16 CCV, DELTA = 20.

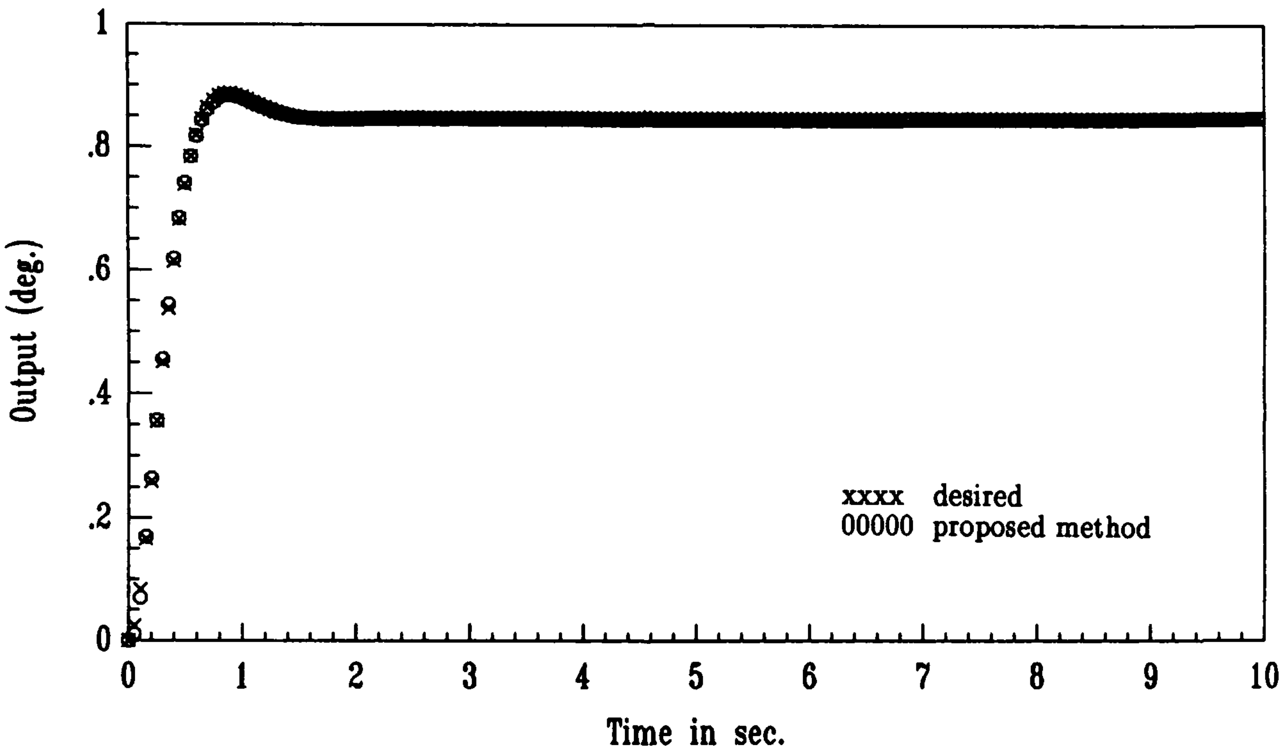

Figure 5.36: Time response of the (2,1) element of the compensated system and the 'desired' system for YF–16 CCV, DELTA = 25.

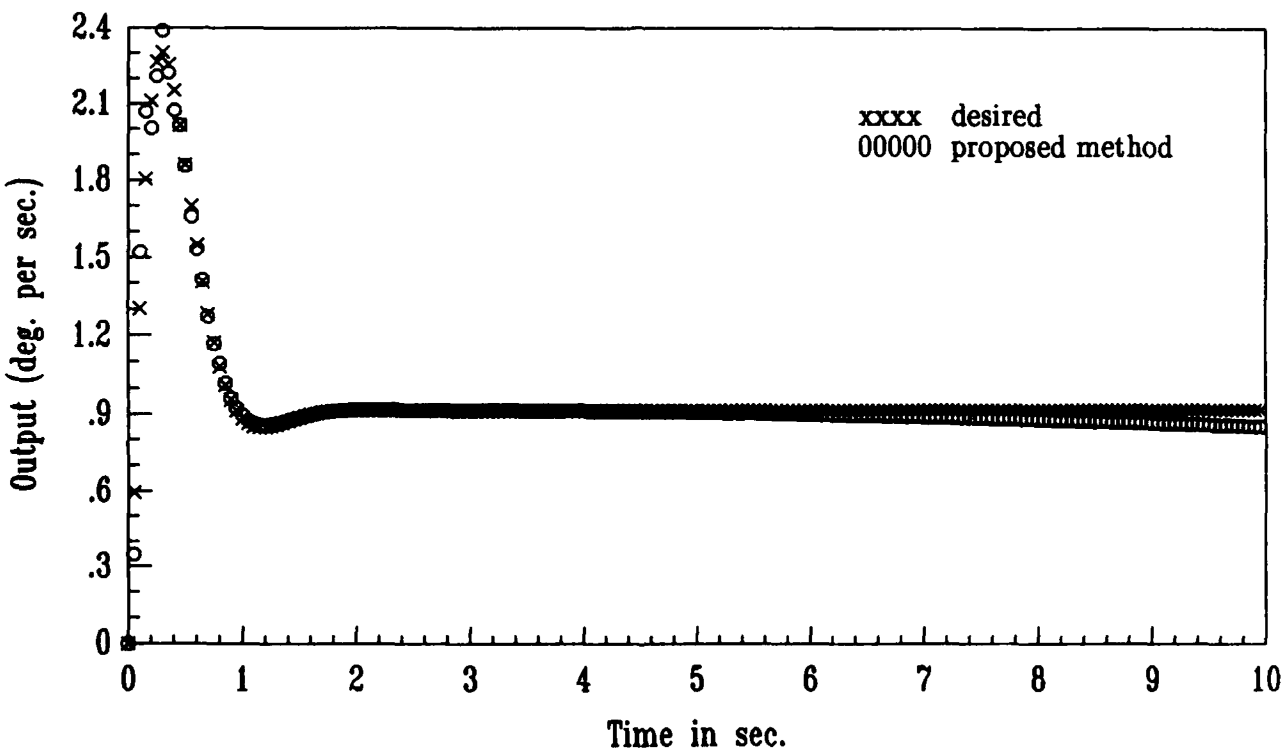

Figure 5.37: Time response of the (2,2) element of the compensated system and the 'desired' system for YF-16 CCV, DELTA = 25.

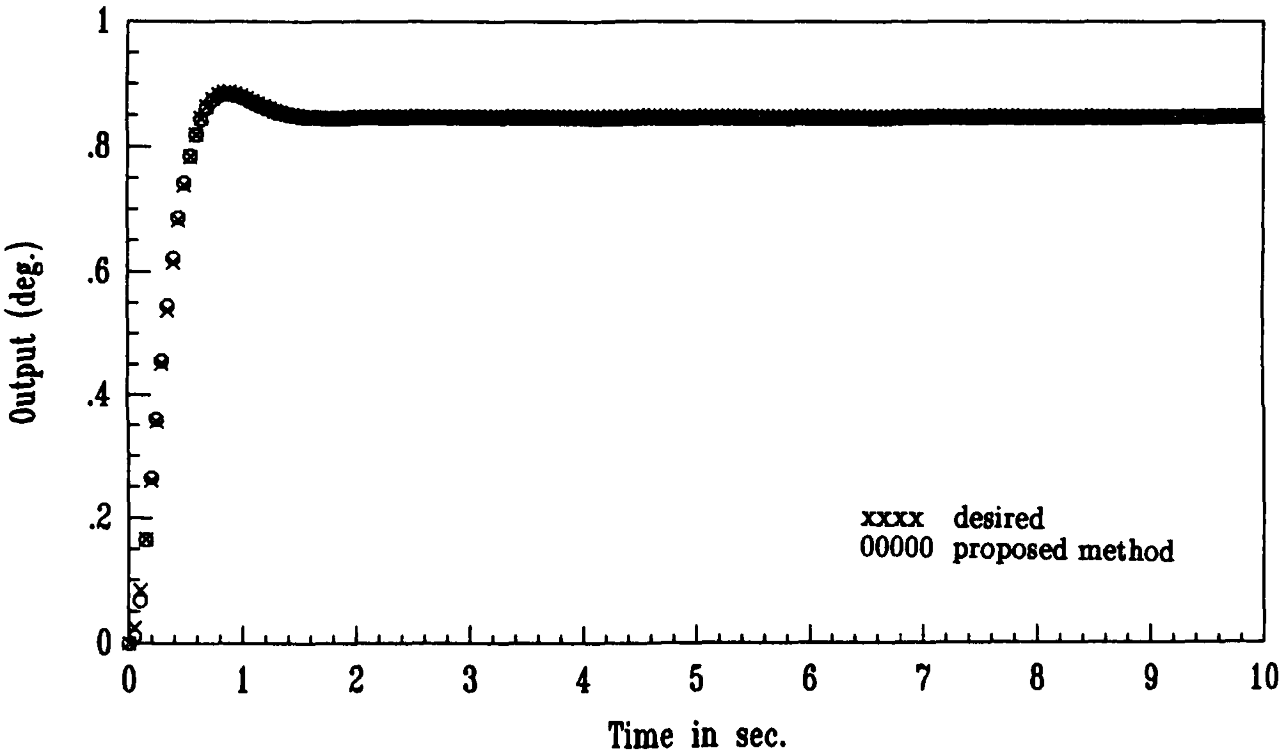

Figure 5.38: Time response of the (1,1) element of the compensated system and the 'desired' system for YF-16 CCV, DELTA = 30.

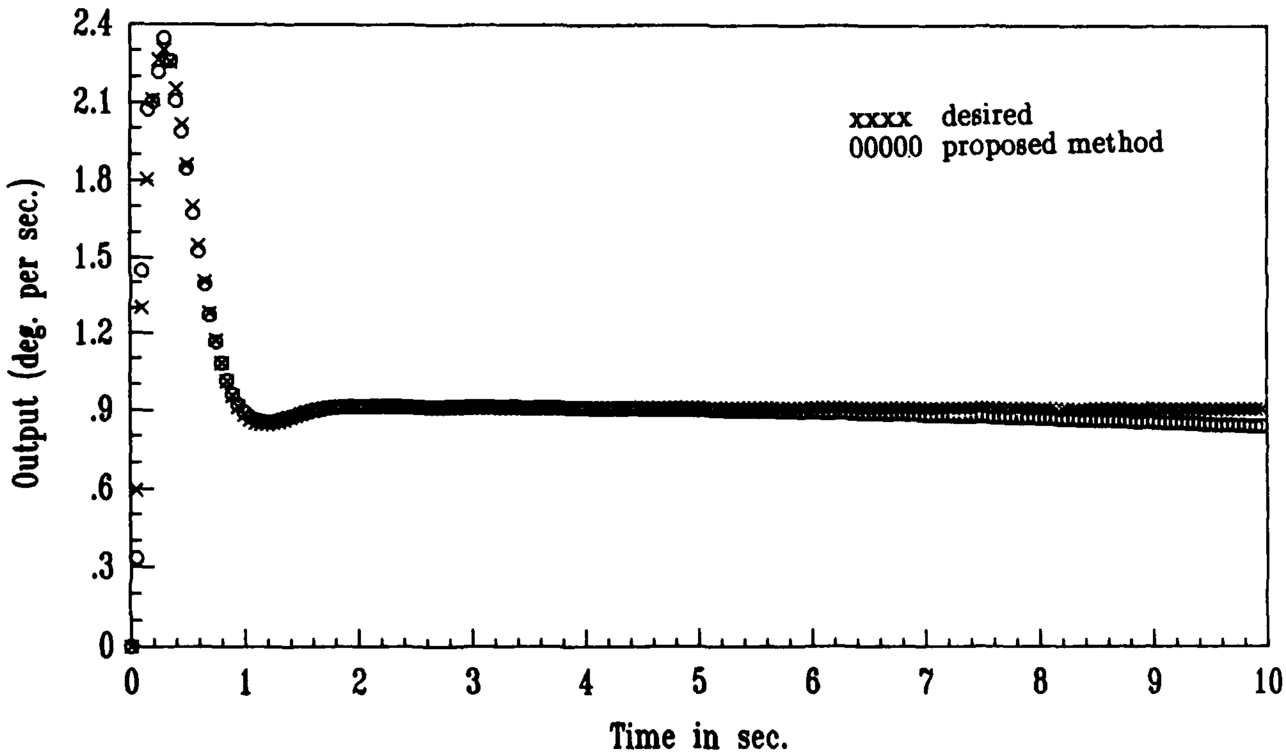

Figure 5.39: Time response of the (2,1) element of the compensated system and the 'desired' system for YF–16 CCV, DELTA = 30.

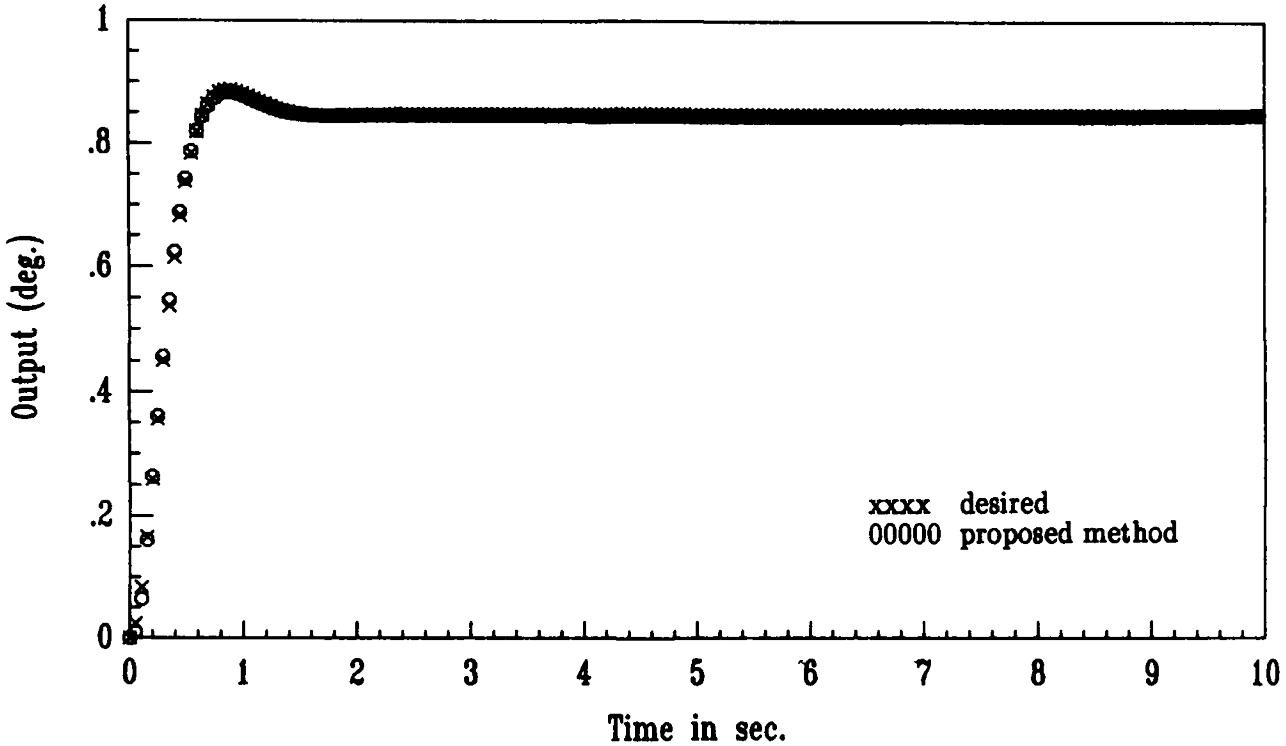

Figure 5.40: Time response of the (1,1) element of the compensated system and the 'desired' system for YF–16 CCV, DELTA = 34.

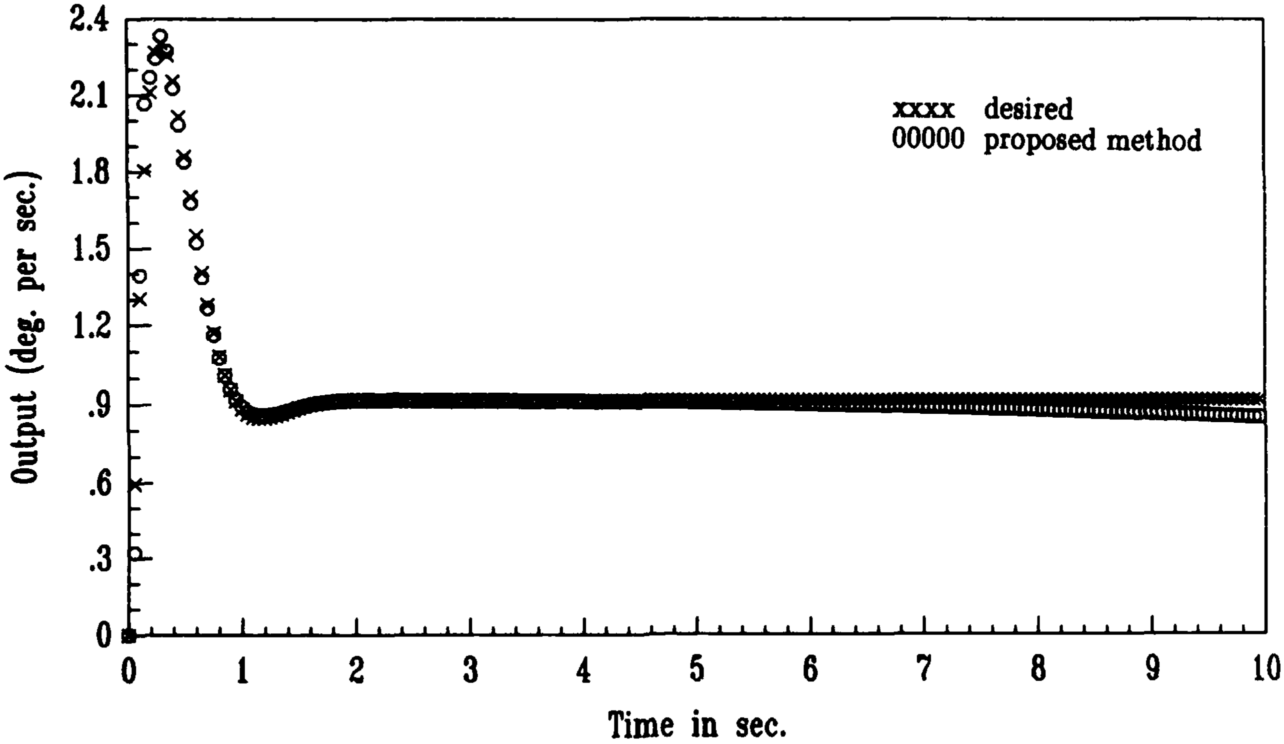

Figure 5.41: Time response of the (2,1) element of the compensated system and the 'desired' system for YF-16 CCV, DELTA = 34.

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